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Integrable PDEs arising as commutation of vector fields: Cauchy problem, longtime behaviour, particular solutions and multidimensional wave breaking

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It was observed long ago [1] that the commutation of multidimensional vector fields can generate integrable nonlinear partial differential equations (PDEs) in arbitrary dimensions. Some of these equations are dispersionless (or quasi-classical) limits of integrable PDEs, having the dispersionless Kadomtsev - Petviashvili (dKP) as universal prototype example, they arise in various problems of Mathematical Physics and are intensively studied in the recent literature. Distinguished examples of PDEs arising as the commutation conditions $[\hat{L}_1(\lambda), \hat{L}_2(\lambda)] = 0$ of pairs of one parameter families of vector fields, being $\lambda \in \mathbb{C}$ the spectral parameter, are the following. i) The vector nonlinear PDE in $N+4$ dimensions ($N \in \mathbb{N}_+$) [2]:

$$\vec{U}_{t_1 z_2} - \vec{U}_{t_2 z_1} + \left(\vec{U}_{z_1} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_2} - \left(\vec{U}_{z_2} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_1} = \vec{0}, \tag{1}$$

where $\vec{U}(t_1, t_2, z_1, z_2, \vec{x}) \in \mathbb{R}^N$, $\vec{x} \in \mathbb{R}^N$ and ii) its divergenceless and dimensional reduction: $\theta_{tx} - \theta_{zy} + \theta_{xx}\theta_{yy} - \theta_{xy}^2 = 0$, the celebrated second heavenly equation of Plebanski, describing self-dual vacuum solutions of the Einstein equations. iii) The following system of two nonlinear PDEs in $2+1$ dimensions [3]:

$$\begin{array}{l} u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} = 0, \\ v_{xt} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} = 0, \end{array} \tag{2}$$

describing a general integrable Einstein-Weyl metric, and its $v = 0$ and $u = 0$ reductions, the celebrated dKP equation $(u_t + uu_x)_x + u_{yy} = 0$ (describing the evolution of small amplitude, nearly one-dimensional waves in shallow water near the shore, when the x-dispersion can be neglected, as well as unsteady motion in transonic flow and nonlinear acoustics of confined beams) and the Pavlov equation $v_{xt} + v_{yy} = v_y v_{xx} - v_x v_{xy}$, associated with non Hamiltonian vector fields. iv) The two-dimensional dispersionless Toda (2ddT) equation $\phi_{\zeta_1\zeta_2} = \left(e^{\phi_t}\right)_t$ (or $\varphi_{\zeta_1\zeta_2} = (e^{\varphi})_{tt}$, $\varphi = \phi_t$), describing integrable heavens and Einstein - Weyl geometries, whose string equations solutions are relevant in the ideal Hele-Shaw problem. The Inverse Spectral Transform (IST) for 1-parameter families of multidimensional vector fields, developed in [2], has allowed one to construct the formal solution of the Cauchy problem for the heavenly equation in [2], for the system (2) and for the dKP equation in [3], for Pavlov equation in [4] and for the wave form $(e^{\phi_t})_t = \phi_{xx} + \phi_{yy}$ of the 2ddToda equation in [5]. This IST turns out to be an efficient tool to study several properties of the solution space of the PDE under consideration: i) the characterization of a distinguished class of spectral data for which the associated nonlinear RH problem is linearized, corresponding to a class of implicit solutions of the PDE; ii) the construction of the longtime behaviour of the solutions of the Cauchy problem [6, 5]; iii) the possibility to establish whether or not the lack of dispersive terms in the nonlinear PDE causes the breaking of localized initial profiles (for the dKP and 2ddT equations respectively in [6] and in [5]) and, if yes, to investigate in a surprisingly explicit way the analytic aspects of such a wave breaking [6]. In this talk we review some of the aspects of this novel theory.

References

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