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Dynamical systems that are approximated by solitons - Two-component decomposition of solutions

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1 General remarks

In multiple-soliton solutions of integrable nonlinear evolution equations, the solitons collide elastically. Far from the finite collision region, each soliton may be affected by the other solitons only through a possible phase shift. Often, these evolution equations provide approximations to more complex dynamical systems. The local nature of soliton interactions can be extended to the perturbative construction of solutions for a wide variety of such complex systems.

To this end, one has to construct the perturbative solution of a dynamical system as a sum of terms that represent elastic and inelastic soliton collisions. These include a plethora of inelastic processes, e.g., soliton-anti-soliton "exchange", soliton-anti-soliton annihilation or creation, soliton merging and decay, and more. Difficulties encountered in traditional expansion procedures do not arise in this two-component approach.

One first solves the equations of the dynamical system in the single-soliton case. uS, the zero-order approximation to the solution, is a single-soliton solution of the integrable Normal Form associated with the dynamical system. The higher-order corrections in the solution can be solved for in closed form as differential polynomials in uS (polynomials in uS and its spatial derivatives).

Replacing uS by uM, a multiple-soliton solution of the Normal Form, everywhere in the full single-soliton solution described above, yields the elastic component of the solution of the complex system in the multiple-soliton case. All contributions in this component represent elastic soliton scattering processes. Through any finite order, they do not induce changes in soliton wave numbers and velocities.

The remainder of the perturbation, above the part that generates the elastic component in the solution, represents inelastic interactions and generates the inelastic component. By construction, inelastic interactions are genuine multiple-soliton effects. The inelastic driving terms as well as the inelastic component in the solution vanish identically in the single-soliton case. Thanks to this property, order-by-order, inelastic driving terms are localized in the soliton collision region, and fall off exponentially in all directions in the x - t plane.

Some inelastic driving terms generate contributions to the solution, which can be expressed in closed-form (differential polynomials in uM). Others (obstacles to asymptotic integrability) generate contributions that have to be found numerically.

In traditional expansions, obstacles to asymptotic integrability either generate unbounded contributions in higher-order corrections the solution, or spoil the simple elastic-scattering structure of the zero-order approximation. These difficulties do not arise in the two-component approach. The zero-order term remains a multiple-soliton solution of an integrable Normal Form, and all higher-order contributions are bounded. Away from the soliton collision region, the inelastic component in the solution tends into a sum of single-solitons. The amplitude of each soliton is determined by the localized perturbation, and is affected by the other solitons. The only additional effect of obstacles to asymptotic integrability is the generation of dispersive waves that decay away from soliton trajectories. As in the elastic component, through any finite order, the inelastic component does not modify soliton wave numbers and velocities.

The two-component approach has been employed in the construction of solutions of the perturbed KdV equation with a general perturbation and the particular case of a Sawada-Kotera perturbation (both through third order), the perturbed NLS equation (through second order), and the perturbed mKdV equation (through first order). It has been also employed (through third order) in the analysis of the Shallow-Water problem and the ion acoustic wave equations of Plasma Physics. The latter application will be reviewed.

Potentially, the two-component approach ought to be applicable to the perturbative analysis of solutions of other dynamical systems in (1+1) or higher dimensions, and simplify it. What is required is that these systems can be approximated by integrable evolution equations, which are solved by localized structures that interact only within a finite domain in space and time. These may include systems, the solutions of which can be approximated by multiple-compacton or peakon solutions, and of pattern-formation phenomena involving structures that are localized in space and time, such as spikes generated in a fluid- or granular-material-layer under the effect of vibrations.

References

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