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The discrete modified Novikov-Veselov hierarchy and discrete differential geometry

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1 Introduction

The modified Novikov-Veselov equation

$$u_t = \left(u_{zzz} + 3u_z v + \frac{3}{2}uv_z\right) + \left(u_{\bar{z}\bar{z}\bar{z}} + 3u_{\bar{z}} v + \frac{3}{2}uv_{\bar{z}}\right), \quad v_{\bar{z}} = (u^2)_z,$$

is an integrable equation that has in recent years attracted significant attention due to its relationship with the classical problem of describing two-dimensional surfaces in three-dimensional space [1]. This equation describes deformations of the Dirac operator preserving its zero-energy level, i.e. isospectral deformations of the equation

$$D\,\psi = \begin{pmatrix} u & \partial \\ -\bar{\partial} & u \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0.$$

The Weierstrass representation associates to each solution of the Dirac equation a surface in three-dimensional space, and any surface locally admits such a description. This description has recently been proposed as a possible approach to the Willmore conjecture.

Discrete differential geometry is a new field which aims to find the discrete equivalents of notions and methods of smooth surface theory [2]. Recently, the discrete analogues of many interesting geometries associated with integrable systems have been found.

We consider the problem of constructing an integrable discretization of the Dirac operator and the associated modified Novikov-Veselov hierarchy, and of finding the correct discrete analogue of the Weierstrass representation. The standard method consists in first discretizing the auxiliary spectral data and the linear operators. The spectral data associated to the differential Dirac operator is an algebraic curve with two marked points and exponential singularities at those points. To construct the discretization, we replace each exponential singularity with a pair of meromorphic singularities. We find that the appropriate discrete analogue of the differential Dirac operator is the following difference operator

$$T_2\psi_1 = \alpha\psi_1 + \beta\psi_2$$
$$T_1\psi_2 = \beta\psi_1 + \alpha\psi_2$$

where α and β are functions of two discrete variables n and m satisfying the condition

$$\alpha^2 - \beta^2 = 1.$$

Proceeding in the same way as for the differential modified Novikov-Veselov equation [3], we construct the discrete modified Novikov-Veselov hierarchy in the form of L, A, B triples. We present the first two equations of the hierarchy and show that they converge in the continuous limit to the corresponding differential equations.

Finally, we show how to associate a discrete surface in three-dimensional space to any solution of the discrete Dirac equation, and consider the geometric properties of the obtained lattices.

References

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[3] I. A. Taimanov "Finite-gap solutions of the modified Novikov-Veselov equations: their spectral properties and applications" *Siber. Math. J.*, **40**, 6, 1999