NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems
Isola Rossa, May 16–23, 2009.

A Generalized Dressing Approach for Solving the Extended KP and the Extended mKP Hierarchy

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February 12, 2009

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1 Generalized dressing approach for exKPH

Let $L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots$, $B_n = L^n_{\geq 0}$, the KP hierarchy reads[1]

$$L_{t_n} = [B_n, L].$$

Recently, inspired by the squared eigenfunction symmetry constraint, we proposed an approach to construct an *extended KP hierarchy* (exKPH) [2]

$$\partial_{\tau_k} L = [B_k + \sum_{i=1}^{N} q_i\partial^{-1}r_i, L], \quad N \geq 0 \tag{1a}$$

$$\partial_{t_n} L = [B_n, L], \quad \forall n \neq k \tag{1b}$$

$$\partial_{t_n} q_i = B_n(q_i),\ \partial_{t_n} r_i = -B_n^*(r_i), \quad i = 1, \ldots, N. \tag{1c}$$

Under (1b) and (1c), showing the commutativity of (1a) and (1b) leads to the zero curvature equation for exKPH (1) and its Lax pair

$$B_{n,\tau_k} - B_{k,t_n} + [B_n, B_k] - \sum_{i=1}^{N}[q_i\partial^{-1}r_i, B_n]_{\geq 0} = 0 \tag{2a}$$

$$q_{i,t_n} = B_n(q_i),\ r_{i,t_n} = -B_n^*(r_i), \quad i = 1, \ldots, N. \tag{2b}$$

$$\Psi_{t_n} = B_n(\Psi), \quad \Psi_{\tau_k} = (B_k + \sum_{i=1}^{N} q_i\partial^{-1}r_i)(\Psi). \tag{3}$$

(2) contains the first and second type of KP equation with self-consistent source. Assume that L of exKPH can be written in the dressing form: $L = W\partial W^{-1}$ [3]. In this paper, we propose a method to generalize the dressing approach for the KPH to exKPH.

Proposition 1. Let f_i, g_i satisfy

$$\partial_{t_n} f_i = \partial^n(f_i), \quad \partial_{\tau_k} f_i = \partial^k(f_i), \tag{4a}$$

$$\partial_{t_n} g_i = \partial^n(g_i), \quad \partial_{\tau_k} g_i = \partial^k(g_i) \quad (i = 1, \ldots, N). \tag{4b}$$

$$h_i = f_i + \alpha_i(\tau_k) g_i \quad i = 1, \ldots, N, \tag{5}$$

$$W = \frac{1}{Wr(h_1, \cdots, h_N)} \begin{vmatrix} h_1 & h_2 & \cdots & h_N & 1 \\ h_1' & h_2' & \cdots & h_N' & \partial \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_1^{(N)} & h_2^{(N)} & \cdots & h_N^{(N)} & \partial^N \end{vmatrix} \tag{6}$$

$$q_i = -\dot{\alpha}_i W(g_i) \quad r_i = (-1)^{N-i} \frac{Wr(h_1, \cdots, \hat{h}_i, \cdots, h_N)}{Wr(h_1, \cdots, h_N)}, i = 1, \ldots, N \tag{7}$$

then W, L, q_i, r_i satisfy (1).

2 exmKPH, reduction and gauge transformation

Inspired by the squared eigenfunction symmetry constraint [4], we propose
Proposition 2. The *extended mKP hierarchy* (exmKPH) is constructed as follows

$$L_{\tau_k} = [B_k + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial, L], \tag{8a}$$

$$L_{t_n} = [B_n, L], \quad n \neq k, \tag{8b}$$

$$q_{i,t_n} = B_n(q_i), \qquad r_{i,t_n} = -(\partial B_n \partial^{-1})^*(r_i),\ i = 1, \cdots, N. \tag{8c}$$

The t_n*-reduction* of the exmKPH leads to

$$\mathcal{L}_{\tau_k} = [\mathcal{L}_{\geq 1}^{k/n} + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial, \mathcal{L}] \tag{9a}$$

$$\lambda_i^n q_i = \mathcal{L}(q_i),\ \lambda_i^n r_i = -\partial^{-1} \mathcal{L}^* \partial(r_i), \quad i = 1, \ldots, N \tag{9b}$$

where $\mathcal{L} = L^n = \partial^n + V_{n-2}\partial^{n-1} + \cdots + V_0 \partial$. The τ_k reduction leads to

$$(L_k)_{t_n} = [(L_k)_{\geq 1}^{n/k}, L_k], \tag{10a}$$

$$q_{i,t_n} = B_n(q_i),\ r_{i,t_n} = -(\partial^{-1} B_n^* \partial)(r_i) \tag{10b}$$

where $L_k = L_{\geq 1}^k + \sum_i q_i \partial^{-1} r_i \partial$, $B_n = (L_k)_{\geq 1}^{n/k}$.
Proposition 3. Suppose L, q_i, r_i satisfy (1), f is a solution of Lax pair (3), then

$$\tilde{L} := f^{-1} L f, \quad \tilde{q}_i := f^{-1} q_i, \quad \tilde{r}_i := -D^{-1}(f r_i)$$

satisfies the exmKPH (8).

References

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