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Yang-Baxter maps and integrability

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This lecture is a review of the theory of Yang-Baxter maps in relation with integrable systems. The main idea is that a reversible Yang-Baxter map is one more feature of integrable hierarchies, which has a very convenient dynamical meaning and is as informative as the usual Lax pair.

Let X be any set. A map $R : X \times X \to X \times X$ is called *Yang-Baxter map* if it satisfies the Yang-Baxter relation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{1}$$

where $R_{ij} : X^3 \to X^3$ is the map acting as R on i-th and j-th factors and identically on the others. Physicists may think of X as the space of internal parameters of the interacting particles and R as the change due to interaction. The Yang-Baxter relation means that the result of interaction of several particles is *independent of the order of interactions:*

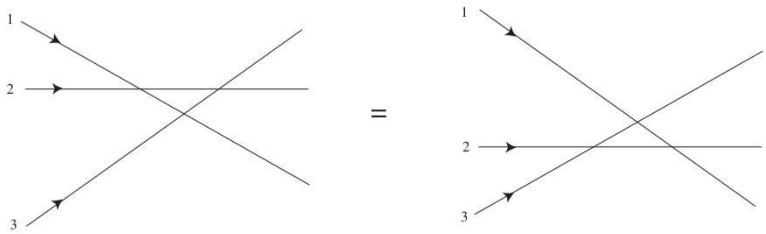

Figure 1: Yang-Baxter relation

This property is also closely related to the *consistency around a cube*, which can be considered as a definition of integrability of the discrete equations on the lattice (see [1, 2]).

R is called *reversible Yang-Baxter map* if it additionally satisfies the relation

$$R_{21}R = Id. \tag{2}$$

The *transfer maps* $T_i^{(n)} : X^n \to X^n, i = 1, \ldots, n$ are defined as

$$T_i^{(n)} = R_{ii+n-1}R_{ii+n-2}\ldots R_{ii+1}, \tag{3}$$

where the indices are considered modulo n.

It can be shown that for any reversible Yang-Baxter map R the transfer maps $T_i^{(n)}, \quad i = 1, \dots, n$ commute with each other:

$$T_i^{(n)} T_j^{(n)} = T_j^{(n)} T_i^{(n)}, \tag{4}$$

and satisfy the property

$$T_1^{(n)} T_2^{(n)} \dots T_n^{(n)} = Id. \tag{5}$$

Moreover, these two properties characterize the reversible Yang-Baxter maps.

Matrix $A(x, \zeta)$ is called the *Lax matrix* of a parameter-dependent Yang-Baxter map $R(\lambda, \mu)$ if the relation

$$A(x, \lambda; \zeta) A(y, \mu; \zeta) = A(\tilde{y}, \mu; \zeta) A(\tilde{x}, \lambda; \zeta), \tag{6}$$

is satisfied iff $(\tilde{x}, \tilde{y}) = R(\lambda, \mu)(x, y)$. Very often the Lax matrix *can be extracted directly* from the Yang-Baxter map. The transfer-dynamics preserve the spectrum of the corresponding *monodromy matrix*

$$M(x_1, \dots, x_n; \zeta) = A(x_n, \zeta) A(x_{n-1}, \zeta) \dots A(x_1, \zeta),$$

which gives the integrals in the standard way. The theory of Poisson Lie groups can be used to explain the Hamiltonian properties of this dynamics, thus completing the integrability picture behind the reversible Yang-Baxter map.

A good example is the *Adler map*:

$$\tilde{x} = y - \frac{\lambda - \mu}{x + y} \qquad \tilde{y} = x - \frac{\mu - \lambda}{x + y}, \tag{7}$$

which gives rise to the KdV hierarchy. There are also interesting examples of Yang-Baxter maps coming from the theory of matrix solitons, geometry, representation theory and combinatorics (see a review [2], where also many relevant references can be found).

The talk is largely based on the results found in collaboration with Goncharenko, Papageorgiou, Reshetikhin, Suris and Tongas.

References

[1] Adler,V.E., Bobenko, A.I., Suris, Yu.B.: Classification of integrable equations on quad-graphs. The consistency approach. *Comm. Math. Phys.* **233** (2003) 513-543.

[2] Veselov, A.P.: Yang-Baxter maps: dynamical point of view. *MSJ Memoirs.* **17** (2007) 145-167.