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One-phase averaging for Ablowitz-Ladik system

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The main object of the study is Ablowitz-Ladik system

$$-i\dot{q}_n - (1 - q_n r_n)(q_{n-1} + q_{n+1}) + 2q_n = 0$$
$$-i\dot{r}_n + (1 - q_n r_n)(r_{n-1} + r_{n+1}) - 2r_n = 0,$$

($n \in \mathbf{Z},\ q_n(t),\ r_n(t),\ t \in \mathbf{C}$ are unknown sequences) which under condition $r_n = \pm\overline{q}_n$ transits to scalar NLS discrete analogue

$$i\partial_t A_n + (1 \pm |A_n|^2)(A_{n+1} + A_{n-1}) - 2A_n = 0.$$

There exists a complete set of the S-integrability attributes for this system:

1) zero curvature representation :

$$-i\partial_t L_n + L_n B_n - B_{n+1} L_n = 0, \qquad (1)$$

with certain matrices L, B;

2) Hamiltonian structure, commuting conservation laws;

3) Algebrogeometric, analytical picture, finite-genus solutions in theta-functions.

The goal is to develop averaging procedure over one-phase solutions that may be written in terms of elliptic functions:

$$q_n = [\zeta(z) - \zeta(z - U) + c]e_n(t),$$
$$r_n = a[\zeta(z + b) - \zeta(z + b - U) + c]e_n^{-1}(t),$$

$z = Un + itV + w,\ e_n(t) = \exp(U_0 n + itV_0)$ with Weierstrass zeta ζ, and study some solutions to the averaged system.

Theorem. The averaged system is Hamiltonian with respect to certain hydrodynamic Poisson bracket.

The Ablowitz-Ladik chain is compatibility condition for the following linear problem:

$$U_{n+1}(t, z) = L_n(t, z)U_n(t, z)$$
$$-i\partial_t U_n(t, z) = B_n(t, z)U_n(t, z),$$

and $U_n = (u_n^1, u_n^2)^T$ is unique on Riemann surface $\Gamma(y, z)$ genus g ($g = 0, 1, ...$):

$$U_n = U_n(t, P),\ P \in \Gamma(y, z) :\ y^2 = \prod_{j=1}^{2g+2} (z - z_j),$$

$z_j,\ j=1,2,...,2g+2$ are branch points. On $\Gamma(y,z)$ one can determine Abelian differentials:

$$\omega_{(2)}=\frac{i}{2}\left[1+\frac{1}{z^2}++\frac{z^{g+1}+\frac{\sqrt{\xi}}{z^2}-\frac{\sqrt{\xi}}{2z}\chi-\frac{z^g}{2}\Lambda+P^{(2)}_{g-1}(z)}{y}\right]dz,$$

$$\omega_{(3)}=\frac{1}{2}\left[\frac{1}{z}+\frac{z^g-\frac{\sqrt{\xi}}{z}+P^{(3)}_{g-1}(z)}{y}\right]dz,$$

$\xi=\prod_{j=1}^{2g+2}z_j$, $\Lambda=\sum_{j=1}^{2g+2}z_j$, $\chi=\sum_{j=1}^{2g+2}\frac{1}{z_j}$, $y^2=\prod_{j=1}^{2g+2}(z-z_j)$, polynomials $P^{(j)}_{g-1}(z)$ provide norming constraints:

$$\oint_{a_j}\omega_{(2)}=\oint_{a_j}\omega_{(3)}=0,\ j=1,2,...,g.$$

The Ablowitz-Ladik chain is reduced to scalar form under condition that set of branch points is invariant over involution $z\to\bar{z}^{-1}$ (see [1]). In the "‘focusing"’ case $((r_n(t)=-\bar{q}_n(t))$ all the branch points lie outside of unit circle. In this case the modulation equation is

$$\partial_T\omega_{(3)}=\partial_X\omega_{(2)},$$

or

$$\partial_T z_j+c(z_j,\mathbf{z})\partial_X z_j=0,\quad j=1,2,...,2g+2,\ \mathbf{z}=(z_1,z_2,...,z_{2g+2}),$$

$$c(z,\mathbf{z})=-i\frac{z^{g+1}+\frac{\sqrt{\xi}}{z^2}-\frac{\sqrt{\xi}}{2z}\chi-\frac{z^g}{2}\Lambda+2P^{(2)}_{g-1}(z,\mathbf{z})}{-\frac{\sqrt{\xi}}{z}+z^g+2P^{(3)}_{g-1}(z,\mathbf{z})}.$$

One can express the norming constants $P_0^{(2,3)}(z,\mathbf{z})$ via standard complete elliptic integrals:

$$P_0^{(2)}=\left(\oint_a\frac{dz}{y}\right)^{-1}\left(\frac{\chi\sqrt{\xi}}{2}\oint_a\frac{dz}{zy}+\frac{\Lambda}{2}\oint_a\frac{zdz}{y}-\oint_a\frac{z^2dz}{y}-\sqrt{\xi}\oint_a\frac{dz}{z^2y}\right),$$

$$P_0^{(3)}=\left(\oint_a\frac{dz}{y}\right)^{-1}\left(\sqrt{\xi}\oint_a\frac{dz}{zy}-\oint_a\frac{zdz}{y}\right),$$

where

$$\oint_a\frac{dz}{y}=(z_3-z_2c^2)K(k)-\frac{z_2c^2(1-c^2)}{k^2}[K(k)-E(k)],$$

$$\oint_a\frac{zdz}{y}=(z_3-z_2)\left\{(z_3-z_2)\Pi_1(c^2,k)-\frac{z_2^2c^2}{(z_4-z_2)k^2}[K(k)-E(k)]\right.$$
$$\left.+z_2\left(1+\frac{z_2}{z_4-z_2}\right)K(k)\right\},\ c^2=\frac{z_4-z_3}{z_4-z_2},\ k^2=c^2\frac{z_1-z_2}{z_1-z_3},$$

and so on with other formulas. K,E,Π are standard complete elliptic integrals. Numerical investigation of self-similar solutions to the averaged system allows to formulate Gurevich-Pitaevsky-type hypotheses on their applicability to developing asymptotic pictures for solutions to various Cauchy problems for the initial Ablowitz-Ladik chain.

[1] P.D. Miller, N.M. Ercolani, I.M. Krichever and C.D. Levermore. Finite genus solutions to the Ablowitz-Ladik equations. Comm. Pure Appl. Math., v. 48, p. 1369-1440, (1995)