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Time discretizations of integrable lattices: local equations of motion

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We discuss the problem of integrable full-discretization of differential-difference equations (lattices) in $1+1$ dimensions. The continuous independent variable to be discretized is regarded as time. The problem of time discretization is, by its nature, distinct from the problem of space discretization. In fact, the former problem has its own peculiarities and difficulties that the latter does not have. This point was uncovered by Ablowitz and Ladik [1] in their attempt to fully discretize the space-discretized nonlinear Schrödinger (NLS) equation (so-called Ablowitz–Ladik lattice). It turned out that unexpected nonlocality emerges in the stage of time discretization; the full-discrete NLS equation involves infinite sums and/or infinite products with respect to the discrete spatial variable and thus is a global-in-space scheme. The full-discrete NLS equation can superficially be written in a local form using additional dependent variables called *auxiliary variables*, but it does not provide any essential resolution of the nonlocality problem.

We propose a *systematic* method for discretizing the time variable in integrable lattices maintaining *locality* of the equations [2]. In contrast to the other known methods [3], our method generally requires no *ad hoc* treatment on a case-by-case basis and appears to have no serious limitations in its applicability; it can be applied to possibly all integrable lattices in $1+1$ dimensions possessing a Lax-pair representation. In particular, it can be used to obtain local full-discretizations of the NLS-type lattices, including the Ablowitz–Ladik lattice. Actually, our method can be considered the *completed version* of the Ablowitz–Ladik approach [1]; it both refines and extends their work in an essential way. A decisive breakthrough can be made by using the lowest-order “conservation laws” derived from the zero-curvature condition. The requirement that all the fluxes corresponding to the same conserved density have to coincide up to a trivial difference results in an “ultralocal” algebraic system for the auxiliary variables. Thus, by solving this algebraic system, we can restore the locality of the equations; that is, the global terms appearing in the stage of time discretization can be replaced by *local* expressions in terms of the original dependent variables. The time-discretized lattices have the same set of conserved quantities and the same structures of solutions as those of the time-continuous lattices; only the time evolution of the parameters in the solutions that correspond to the angle variables is discretized.

As an example, we consider the time discretization of the (non-reduced) Ablowitz–Ladik lattice

$$\begin{cases} \dfrac{1}{h}\left(\widetilde{q}_n - q_n\right) + (1 - q_n r_n)\Lambda_{n+1}\left(c\widetilde{q}_{n+1} - dq_{n-1}\right) - \alpha q_n + \beta \widetilde{q}_n = 0, \\ \dfrac{1}{h}\left(\widetilde{r}_n - r_n\right) + (1 - q_n r_n)\Lambda_{n+1}\left(d\widetilde{r}_{n+1} - cr_{n-1}\right) - \beta r_n + \alpha \widetilde{r}_n = 0, \\ \left(1 - \widetilde{q}_n \widetilde{r}_n\right)\Lambda_n = (1 - q_n r_n)\Lambda_{n+1}, \qquad \lim\limits_{n\to-\infty} \Lambda_n = 1 \text{ or } \lim\limits_{n\to+\infty} \Lambda_n = 1, \end{cases} \tag{1}$$

where the tilde denotes the forward shift $(m \to m+1)$ in the discrete time coordinate $m \in \mathbb{Z}$, and h is a parameter usually interpreted as the difference interval of time. The decaying boundary conditions $\lim_{n\to\pm\infty} q_n = \lim_{n\to\pm\infty} r_n = 0$ are assumed. It is *redundant* to impose the conditions $\lim_{n\to\pm\infty} \Lambda_n = 1$ at both spatial ends and it is nontrivial (though verifiable) that the redundant conditions are compatible. Thus, the auxiliary variable Λ_n can be written globally as

$$\Lambda_n = \prod_{j=-\infty}^{n-1} \frac{1 - \widetilde{q}_j \widetilde{r}_j}{1 - q_j r_j} \quad \text{or} \quad \prod_{j=n}^{+\infty} \frac{1 - q_j r_j}{1 - \widetilde{q}_j \widetilde{r}_j}.$$

Our method provides the nontrivial lowest-order "conservation law" (before taking the logarithm),

$$\left(1 - \widetilde{q}_n \widetilde{r}_n\right) f_n = (1 - q_n r_n) f_{n+1},$$

$$f_n := \left(1 + hc\widetilde{q}_n r_{n-1}\Lambda_n + h\alpha\right)\left(1 + hdq_{n-1}\widetilde{r}_n\Lambda_n + h\beta\right) + h^2 cd\left(1 - \widetilde{q}_n \widetilde{r}_n - q_{n-1}r_{n-1}\right)\Lambda_n^2.$$

Comparing this with $\left(1 - \widetilde{q}_n \widetilde{r}_n\right)\Lambda_n = (1 - q_n r_n)\Lambda_{n+1}$ and taking the boundary conditions into account, we obtain

$$\frac{f_n}{\Lambda_n} = \frac{f_{n+1}}{\Lambda_{n+1}} = \cdots = (1 + h\alpha)(1 + h\beta) + h^2 cd.$$

We can easily find the proper solution of this quadratic equation for Λ_n. Substituting it into (1), we arrive at a local time discretization of the (non-reduced) Ablowitz–Ladik lattice. By choosing the parameters appropriately and imposing the reduction of the complex conjugate $r_n = -q_n^*$, we obtain

$$\frac{\mathrm{i}}{\delta}\left(\widetilde{q}_n - q_n\right) + \frac{2\left(1 + |q_n|^2\right)\left(\widetilde{q}_{n+1} + q_{n-1}\right)}{1 + \mathrm{i}\delta\mathcal{C}_n + \sqrt{\left(1 + \mathrm{i}\delta\mathcal{C}_n\right)^2 - 4\delta^2\mathcal{D}_n}} = 0,$$

with $\mathcal{C}_n := q_n \widetilde{q}_{n+1}^* - \widetilde{q}_{n+1} q_n^*$ and $\mathcal{D}_n := (1 + |\widetilde{q}_{n+1}|^2)(1 + |q_n|^2)$. Here, $\delta := h/(1 + h^2)$ is a new parameter satisfying $-1/2 \leq \delta \leq 1/2$.

References

[1] M. J. Ablowitz and J. F. Ladik: Stud. Appl. Math. **55** (1976) 213; **57** (1977) 1.

[2] T. Tsuchida: *A systematic method for constructing local time discretizations of integrable lattices*, Preprint OIQP–09–01 (2009).

[3] Yuri B. Suris: *The Problem of Integrable Discretization: Hamiltonian Approach* (Birkhäuser, Basel, 2003).