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Non-isospectral problems and nonlinear integrable hierarchies in $(2+1)$ dimensions

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1 General remarks

Explicit formulations of linear problems (or Lax representations) with non-isospectral parameters (see, the references[1]-[5] and so on) are presented, which give higher-dimensional versions of cerebrated nonlinear integrable hierarchies[6, 7, 8].

2 The Calogero-Bogoyavlenskii-Schiff equation with non-isospectral parameters

Taking the Calogero-Bogoyavlenskii-Schiff equation as an example the author shall explain the non-isospectral problem. The Calogero-Bogoyavlenskii-Schiff(CBS) equation[6, 9, 10] for $q = q(x, z, t)$:

$$q_t + \frac{1}{4} q_{xxz} + q q_z + \frac{1}{2} q_x \partial_x^{-1} q_z = 0, \tag{1}$$

is given by a compatibility condition of the following differential equations, namely the linear problem for the wave function $\Phi = \Phi(x, z, t)$:

$$\begin{cases} \Phi_x = \mathcal{M}\Phi, \\ \Phi_t = \left(\xi^j \partial_z + \mathcal{N}\right)\Phi, \end{cases} \tag{2}$$

where $j = 2$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and

$$\begin{cases} \mathcal{M} = -i\xi\sigma_3 + q\sigma_+ - \sigma_-, \\ \mathcal{N} = \frac{1}{4}\left(2i\xi\partial_x^{-1} q_z - q_z\right)\sigma_3 + \frac{1}{4}\left(2i\xi q_z - q_{xz} - 2q\partial_x^{-1} q_z\right)\sigma_+ + \frac{1}{2}\left(\partial_x^{-1} q_z\right)\sigma_-. \end{cases}$$

Note here that the spectral parameter ξ of the linear problem (2) is satisfied with $\xi_t - \xi^2 \xi_z = 0$, that is, $\xi = \xi(z, t)$. For $\xi_t \neq 0$, ξ and the linear problem (2) can be respectively called *the non-isospectral parameter* and *the non-isospectral problem.* By a dimensional reduction $z = x$ with $\xi_t = 0$, or *the iso-spectral parameter* ξ, the CBS equation (1) can be reduced to the celebrated KdV equation for $q(x, t)$.

3 Non-isospectral problems of higher-dimensional nonlinear integrable hierarchies

Take 2×2 matrix in the non-isospectral problem (2):

$$\mathcal{M} = iF(\xi)\sigma_3 + G(\xi)q\sigma_+ + G(\xi)r\sigma_-,$$

where $q = q(x,z,t)$, $r = r(x,z,t)$ and being $\xi_t \neq 0$. Here another matrix $\mathcal{N}$ can be constructed suitably by the compatibility condition. Note that

- For $F(\xi) = \xi$ and $G(\xi) = 1$, the non-isospectral problem (2) reads a $(2+1)$ dimensional version of the **Ablowitz-Kaup-Newell-Segur** hierarchy, which gives the CBS equation (1), a modified CBS equation, a higher-dimensional nonlinear Schrödinger equation and their hierarchies.
- For $F(\xi) = \xi^2$ and $G(\xi) = \xi$, the non-isospectral problem (2) reads a $(2+1)$ dimensional version of the **Kaup-Newell** hierarchy, which gives a higher-dimensional derivative nonlinear Schrödinger equation:

  $$iq_t + q_{xz} - i\{q\partial_x^{-1}(qr)_z\}_x = 0, \quad ir_t - r_{xz} - i\{r\partial_x^{-1}(rq)_z\}_x = 0$$

  and its hierarchy.
- For $F(\xi) = G(\xi) = \xi$, , the non-isospectral problem (2) reads a $(2+1)$ dimensional version of the **Wadati-Konno-Ichikawa** hierarchy, which gives a higher-dimensional Harry-Dym equation:

  $$q_t + \frac{1}{4}q^3 q_{xxx} + qq_{xx}q_z - qq_x q_{xz} + q^2 q_{xxz} + q^3 q_{xxx}\partial_x^{-1}\left(\frac{q_z}{q^2}\right) = 0$$

  and its hierarchy.

And exact solutions, conserved quantities and dispersionless versions of their hierarchies may be presented.

References

[1] Burtsev S. P., Zakharov V. E. and Mikhairov A. V.: *Theor. Math. Phys.* **70** (1987) 227.

[2] Bogoyavlenskii O. I.: *Russian Math. Surveys* **45** (1990) 1.

[3] Clarkson P. A., Gordoa P. R. and Pickering A.: *Inv. Prob.* **13** (1997) 1463.

[4] Kobayashi T. and Toda K.: *Symmetry, Integrability and Geometry: Methods and Applications* **2** (2006) paper 063.

[5] Zenchuk A. I. and Santini P. M.: *arXiv:0801.3945.*

[6] Ablowitz M. J., Kaup D. J., Newell A. C. and Segur H.: *Studies in Appl. Math.* **53** (1974) 249.

[7] Kaup D. J. and Newell A. C.: *J. Math. Phys.* **19** (1978) 798.

[8] Wadati M., Konno K., and Ichikawa Y. H.: *Prog. Theor. Phys.* **47** (1979) 1698.

[9] Calogero F.: *Lett. Nuovo Cim.* **14** (1975) 443.

[10] Schiff J.: *NATO ASI Ser. B* **278** (1992) 13pages.