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A uniform reduction method

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1 A uniform reduction method

Many integrable equations are obtained as reductions of larger systems. The fact that this is true for many (integrable) equations of interest in applications makes of the reduction problem one of the central problems in the theory of integrable systems since its early days. A wide class of (algebraic) reductions can be studied in terms of *reduction groups* [1], that is, reductions can be associated with a discrete symmetry group of the corresponding linear problem (Lax Pair). The simplest example of such a symmetry is the conjugation for self-adjoint operators. This purely algebraic reduction technique, first formulated by Mikhailov (see [2]) and later developed in [3], [4], has been successfully applied both in classical (e.g. [5]–[8]) and quantum integrable systems theory (e.g. the classification of solutions of the classical Yang–Baxter equation in [9]). In this communication we report on work in progress on reductions and the search for invariants. In particular we deal with the following problem. Consider 2 by 2 matrices with rational functions of a spectral parameter λ as coeffients. Then GL_2 acts on these by conjugation of the matrices combined with fractional linear transformation of the spectral parameter. By considering representations of finite groups G in GL_2 we induce a representation of G on the above mentioned space of matrices. Associated to G are homogeneous forms that are invariant under the action of G. For instance, if one takes a platonic solid, one can consider all its edges as points (ξ_i, η_i) on the projective line and consider the form

$$f(X,Y) = \prod_i \begin{vmatrix} X & Y \\ \xi_i & \eta_i \end{vmatrix}$$

(cf. Klein [10]). We start by first constructing a GL_2-covariant matrix F (so that it is automatically G-invariant for all G) and using this F in combination with the group dependent f we construct homogeneous G-invariant matrices. It turns out that these contain an $\mathfrak{sl}_2$, a fact that was not at all obvious to us from the start, but which simplifies the analysis considerably, since we can now derive evolution equations from the zero-curvature condition independently of the group. The group seems to plays a role only once one tries to find explicit solutions of the Lax Pair one started with.

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