NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems
Isola Rossa, May 16–23, 2009.

Poles of special solutions to Painlevé I and II equations

Victor Novokshenov[a]

February 24, 2009

a. Institute of Mathematics, Ufa Science Center, Russian Academy of Sciences

All solutions to the Painlevé equations are meromorphic functions. Since the works of P. Painlevé who proved the meromorficity of all solutions of PI equation

$$u'' = 6u^2 - z, \quad z \in \mathbb{C}, \tag{1}$$

the distribution of poles became one of the problems in the analytic theory of nonlinear ODEs.

P. Boutrox [2] was the first who studied the generic PI transcendents and established basic properties of their poles. Unlike general estimates in spirit of Nevanlinna theory, the behavior of Painlevé transcendents is much more regular at infinity, or near essential singularities. This is proved by elliptic function approximations which are valid in appropriate sectors of the complex plane (see [1]).

Iterations of rational solutions to Painlevé equations were discovered long time ago. The regular method to produce these solutions for PII − PVI is based on Bäcklund transformations. They were used in a number of papers to construct families of rational solutions. With the use of Vorob'ev-Yablonskii formulas and Okamoto polynomials P. Clarkson built distributions of poles for PII − PV equation with large coefficients. However, this method cannot give a solution for a fixed set of coefficients, since coefficients are shifted after each Bäcklund transformation. Moreover, this method is useless for PI which has no free coefficients and posesses no rational solutions.

Besides rational solutions, no reliable information is known about poles of Painlevé transcendents in the complex plane for $|z| = O(1)$. To this end, it is natural to apply the (main diagonal) Padé approximation with polynomials P_n, Q_n

$$u(z) \approx \frac{P_n(z)}{Q_n(z)}, \quad P_n(z) - u(z)Q_n(z) = O(z^{2n+1}), \quad z \to 0, \tag{2}$$

to find the solution of equation (1) with given initial data $u(0)$, $u'(0)$. The idea is to avoid traditional Tailor series expansion, but use instead a Bäcklund-type recurrence taking into account the invariance of Painlevé equations under Möbius transformations. This is achieved by a version of the Fair-Luke algorithm [3], which is based on recurrent continuous fraction representation of $u(z)$. It works pretty fast on a PC computer and produces hundreds of poles for a reasonable time. Moreover, since all solutions are meromorphic, this continuous fraction is convergent and gives arbitrary good approximation in any compact domain.

A number of numeric experiments by this method has been done in the paper [5] to study Padé approximate solutions (2) to the Painlevé I and II equations. The distributions of poles for the well-known Ablowitz-Segur and Tracy-Widom solutions to the PII were found. The Boutroux *tritronquée* solution to the PI was shown to have poles only in the critical sector $4\pi/5 < \arg z < 6\pi/5$ of the complex plane z. This gives a numeric confirmation of the Dubrovin's conjecture formulated in [4]. Similarly, all the poles of Tracy-Widom solution to the PII equation $u_{zz} = zu + 2u^3$ are lying in the sectors $\pi/3 < \arg z < 2\pi/3$ and $4\pi/3 < \arg z < 5\pi/3$. The analytic proofs of these statements are given in the present talk.

References

[1] A.S.Fokas, A.R.Its, A.A.Kapaev and V.Yu.Novokshenov, Painlevé Transcendents. The Riemann-Hilbert Approach, Math. Surveys and Monographs, vol. 128, Amer. Math. Soc., Providence, RI, 2006.

[2] P.Boutroux, Ann. École Norm., **30**, 265 - 375 (1913); Ann.École Norm., **31**, 99 - 159 (1914).

[3] W.Fair, Y.Luke, Math.Comp., **20**, 602-605 (1968).

[4] B.Dubrovin, T.Grava and C.Klein, arXiv:0704.0501v2, (2007).

[5] V.Yu.Novokshenov, Theor. Math. Phys. (2009) (to appear).