NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems
Isola Rossa, May 16–23, 2009.

From Darboux-Pöschl-Teller potentials and their deformations to Wronskian and Casorati determinants and Askey-Wilson functions

Vladimir Matveev[a,b]

March 2, 2009

a. Institut de Mathématiques de Bourgogne, Université de Bourgogne, 9 Av. Alain Savary F-21078 Dijon, France.

b. Max-Planck-Instiut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany.

1 General remarks

This contribution, is based on our joint work with Pierre Gaillard. Its content, modulo some additions, can be found in our preprint RIMS-1653, February (2009), 1-18, http://www.kurims-kyoto-u.ac.jp/preprint/file/RIMS1653pdf

We consider some special reductions of generic Darboux-Crum dressing formulas and of their difference versions. As a matter of fact we obtain some new formulas for Darboux-Pöschl-Teller (DPT) equations, (more frequently but not very correctly called Pöschl-Teller, (PT), equations), their difference deformations, and the related eigenfunctions. Our formulas are new even for the most classical case of Darboux-Pöschl-Teller equation labeled by two positive integers m and n. We show that The DPT potentials can be represented by means of Wronskians composed from particular sequence of solutions to the free Schrödinger equation $y'' + k^2 y = 0$, with different values of k. Getting this representation one can conclude, (Frobenius-Crum theorem), that the related eigenfunctions are also given by the ratio of two Wronskians. Wronskian in denominator allows explicit evaluation in trigonometric case being equal to some integer depending on m and n times product of some integer powers of sine and cosine functions with obvious modification for the hyperbolic DPT equation. Comparison of this formulas with the hypergeometric solution to the same equation obtained by Darboux (1882) allows to provide and explicit determinant evaluation formulas for hypergeometric series representing half-integer parameter families of Jacobi functions and Jacobi polynomials. The simple change of variables used already by Darboux links the same determinant formulas with zonal spherical functions on hyperboloids $H^{p,q}$; $p = 2m + 3, q = 2n + 3$. This kind of the result was first derived by Van-Diejen and Kirillov in 2000-2001 on the base of the different representations for the same potentials and their eigenfunctions given by Fredholm determinants. Our formulas have an advantage to provide one-line calculation for the action of the KdV hierarchy on the

DPT initial data. Another advantage of our formulas is that its trigonometric version allows another one-line calculation which proves that any trigonometric DPT potential give rise to the corresponding two-dimensional Huygens potential, whose general structure was described by Berest and Loutsenko (1997) by means of (trigonometric) Wronskian determinants.

Next, we consider two families of difference, (or q-), deformations of the DPT potentials and equations, again labeled by two integers m and n, and the deformation parameter h (or q) which we call respectively DDPT-I potentials, (equations), and DDPT-2 potentials ,(equations). The related eigenfunctions are given always by the ratio of two Casorati determinants. The DDPT-I potentials and DDPT-II potentials are given by elementary expressions being simple rational combinations of exponentials. We show that in the same time they alow representation involving Casorati determinants remarkably composed from the same system of functions as for the Wronskians in the DPT case. Our formulas, being of meromorphic nature, remain valid for the complex values of x,and h. We show that for real values of parameters our potentials in DDPT-II case coincide with particular case of Askey-Wilson potentials and DDPT-II equation can be identified with a special case of Askey-Wilson spectral problem. Here again comparison of our formulas with the related special case of Askey-Wilson q-hypergeometric functions gives new determinant evaluation formulas for these special functions at the values of parameters corresponding to some special reduction of generic 5 parameter Askey-Wilson potential. Advantage of Casoratian representations for the DDPT potentials is of the same nature as in DPT case: it allows one-line calculation of the action of the difference KdV hierarchy on the DDPT-I initial data and of the functional-difference Toda hierarchy on the DDPT-II initial data. For the special case $n = 0$ the lattice version, (i.e. $x \in Z, h = 1$), of DDPT-I equation and of DDPT-II equation and its link with Askey-Wilson polynomials was first found in 1995 by Spiridonov and Zhedanov, who somehow have not obtained the determinant evaluation formulas for the AW-polynomials. Later in 1998 appeared a work by Ruisjenaars (1999) giving some different kind of the solution to the DDPT-I equation with $n = 0$, much more complicated with respect to our results from combinatorial point of view. Soon after Van Diejen and Kirillov (2000) obtained the Fredholm determinant formulas for the solution of generic DDPT-I model with $n = 0$ starting from the reduction of the formulas describing the reflectionless potentials for the difference Schrödinger equation obtained from the standard IST approach. Their work reveals many beautiful combinatorial aspects of the problem in particular the links with combinatorics of Young diagrams. Our approach is based on the ideas of the Darboux transformation method first formulated in 1979, (exactly 30 years ago: see LMP **3**, 214-216, 217-222, 425-429, 503-512 (1979)), by the first contributor in a way extending to linear and nonlinear PDE of any order and their difference versions the remarkable but dispersed results by Frobenius , Darboux and Crum, concerning, (with exception of Frobenius), the linear ODE of the second order. Obviously our results can also be interpreted as providing the new formulas for the eigenfunctions of BC_1 two particle quantum Calogero-Moser hamiltonian and of its difference version.