NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems
Isola Rossa, May 16–23, 2009.

Stäckel systems and solutions of soliton hierarchies.

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February 28, 2009

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1 Introduction

In paper [1]we showed that Stäckel separable systems (i.e. quadratic in momenta Hamiltonian systems separable in the sense of Hamilton-Jacobi theory) give rise to soliton hierarchies. In this talk I intend to show that Stäckel systems yield many interesting solutions of soliton hierarchies. I will concentrate on the case of KdV hierarchy and its multicomponent (coupled) generalizations (Antonowicz-Fordy or cKdV hierarchies [2]). During my talk I will demonstrate (see also [3]) that multiparameter solutions of Stäckel systems lead to new finite-gap, rational and implicit solutions of these hierarchies. In this abstract I focus on the idea of passing from zero-energy solutions of Stäckel systems to rational solutions of cKdV hierarchies.

2 Stäckel systems and Killing systems

Our starting point is a $2n$-dimensional Poisson manifold (M, Π) and a set of Darboux coordinates (λ, μ) for the Poisson operator Π. A set of relations of the form

$$\varphi_i(\lambda_i, \mu_i, a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, n, \;\; a_i \in \mathbf{R}, \quad \det\left(\frac{\partial\varphi_i}{\partial a_j}\right) \neq 0 \qquad (1)$$

(each φ_i depending on λ_i, μ_i only) is called separation relations. Solving it w.r.t. a_i we obtain n functions (Hamiltonians) $a_i = H_i(\lambda, \mu)$ on M yieldning n commuting Hamiltonian systems. Choosing (1) in the form

$$\sum_{j=1}^{n} a_j \lambda_i^{n-j} = \lambda_i^m \mu_i^2 - \lambda_i^k, \quad i = 1, \ldots, n, \quad m, k \in \mathbf{Z}$$

we obtain n quadratic in momenta μ Stäckel Hamiltonians (Stäckel systems) of Benenti type on $M = T^*\mathcal{Q}$

$$H_i = H_i^{n,m,k} = \mu^T K_i G^{(m)} \mu - V_i^{(k)}$$

where $G^{(m)}$ is a metric tensor on a Riemannian manifold $\mathcal{Q}$ and K_r are Killing tensors w.r.t $G^{(m)}$ (for any m).

Theorem 1 *The general $n-$time solution for all Hamilton equations for $H_i^{n,m,k}$ is*

$$t_i + c_i = \pm\frac{1}{2}\sum_{r=1}^{n}\int\frac{\lambda_r^{n-i}}{\sqrt{\lambda_r^m\left(\sum_{j=1}^{n}a_j\lambda_r^{n-j}-\lambda_r^k\right)}}d\lambda_r, \quad i=1,\ldots,n. \qquad (2)$$

To see this it is enough to integrate the related Hamilton-Jacobi problem. Now, with tensors K_i we can associate a set of commuting dispersionless PDE's of evolutionary type (hydrodynamical systems)

$$\lambda_{t_i} = K_i\lambda_x \equiv Z_i(\lambda,\lambda_x), \quad i=1,\ldots,n \qquad (3)$$

that we call dispersionless Killing system. The key observation is the following:

Proposition 2 *Every solution $\lambda(t_1,\ldots,t_n)$ given by (2) solves also (3)*

One proves it by eliminating momenta μ from $\lambda_{t_i} = \frac{\partial H_i}{\partial \mu} = 2AK_iG\mu$. The Hamiltonians $H_i^{n,m,k}$ give rise to n Lagrangians $L_i^{n,m,k}$. Choose now arbitrary natural N and s, put $n = s+N-1$ and consider s n-component Killing system (3) in Viète coordinates $q_i(\lambda)$. These systems have the form

$$q_{t_r} = Z_r^n\left[q_1,\ldots,q_n\right], \;\; r=1,\ldots,s=n-N+1 \qquad (4)$$

where [] denotes differential functions. Consider also the Euler-Lagrange equations $\frac{\delta}{\delta q_i}\left(L_1^{n,-\alpha,2n+N}\right)=0$. It is possible to solve them w.r.t to $q_{N+1},\ldots,q_n$:

$$q_{N+1} = f_1^{(\alpha)}\left[q_1,\ldots,q_N\right],\ldots,q_n = f_{n-N+1}^{(\alpha)}\left[q_1,\ldots,q_N\right] \qquad (5)$$

Equations (2) solve both (4) and (5) so that in the class of solutions (2) we can express $q_{N+1},\ldots,q_n$ as differential functions of $q_1,\ldots,q_N$ in (4). This procedure yields s N-component evolutionary PDE's of the form

$$\overline{q}_{t_r} = \overline{Z}_r^{N,\alpha}\left[\overline{q}\right] \qquad r=1,\ldots,s=n-N+1, \quad \alpha\in\{0,\ldots,N-1\}, \quad \overline{q}=(q_1,\ldots,q_N)^T$$

and an important observation is that if $s\to s+1$ then $n\to n+1$ but the first s systems $\overline{q}_{t_r} = \overline{Z}_r^{N,\alpha}\left[\overline{q}\right]$ remain unaltered. This means that we obtain soliton hierarchies

$$\overline{q}_{t_r} = \overline{Z}_r^{N,\alpha}\left[\overline{q}\right] \qquad r=1,2,\ldots\infty, \quad \left[\overline{Z}_i^{N,\alpha},\overline{Z}_j^{N,\alpha}\right]=0 \text{ all } i,j \qquad (6)$$

one for every choice of α. It can be verified that these are the hierarchies obtained in [2].

Theorem 3 *For any $\beta\in\{0,\ldots,n-1\}$, the n functions $\lambda_i(t_1,\ldots,t_n)$ given implicitly by*

$$t_i + c_i = \pm\frac{1}{2}\sum_{r=1}^{n}\int\frac{\lambda_r^{n-i+\alpha/2-\beta/2}}{\sqrt{\lambda_r^{2n+N-\beta}+\sum_{j=1}^{n}a_j\lambda_r^{n-j}}}d\lambda_r \qquad i=1,\ldots,n \qquad (7)$$

are solutions of the first $n-\beta+1$ eqs of (6)

Proof. Write (2) in our setting. ∎

3 Zero-energy solutions of coupled KdV hierarchies

If $a_i = 0$ for all i (zero-energy case) we can integrate (7) to

$$t_i + c_i = \pm \frac{1}{2 - 2i - \sigma} \sum_{r=1}^{n} \lambda_r^{1-i-\sigma/2} \qquad i = 1, \ldots, n \tag{8}$$

where $\sigma = N - \alpha$. They contain no β and are implicit solutions of the first n systems in (6). They can be solved in Viète coordinates yielding explicit, multiparameter (multi-time) solutions of hierarchies (6).

Example If we take $N = 1 = \sigma + \alpha = 1 + 0$, $s = 3 \Rightarrow n = s + N - 1 = 3$ then our procedure turns the corresponding three Killing systems (4) into first three flows of the KdV hierarchy:

$$q_{1,t_1} = q_{1,x} = \overline{Z}_1^{1,0}, \qquad q_{1,t_2} = \frac{1}{4} q_{1,xxx} + 3q_1 q_{1,x} = \overline{Z}_2^{1,0}$$

$$q_{1,t_3} = \frac{1}{16} q_{1,xxxxx} + \frac{5}{2} q_{1,x} q_{1,xx} + \frac{5}{4} q_1 q_{1,xxx} + \frac{15}{2} q_1^2 q_{1,x} = \overline{Z}_3^{1,0}$$

yielding also their 3-time solution in a rather surprising, rational form

$$q_1(x, t_2, t_3) = \frac{-3\left(675t_3^2 - 270t_3x^5 + 2x^{10} + 675x^4t_2^2 - 1350xt_2^3\right)}{\left(-15t_2x^3 - 45t_2^2 + x^6 + 45xt_3\right)^2}.$$

References

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