

q -deformed KP hierarchy with self-consistent sources

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In recent years, the q -deformed integrable systems attracted many interests both in mathematics and in physics. A very important type of coupled integrable systems is the soliton equation with self-consistent sources, which has many physical applications and can be obtained by coupling some suitable differential equations to the original soliton equation. Very recently, we proposed a systematical procedure to construct a new extended KP hierarchy and its Lax representation [1].

In contrast with the well-studied KdV and KP equation with self-consistent sources, the q -Gelfand-Dickey hierarchy with self-consistent sources and the q -KP hierarchy with self-consistent sources have not been investigated yet. It is interesting to consider the case of the algebra of q -pseudo-differential operator, and to see if our approach could be generalized to construct new extended q -deformed integrable systems, which would enable us to find two types of new q -deformed soliton equation with sources in a systematic way.

In this paper, we will give a systematical procedure to construct a new extended q -deformed KP (q -KP) hierarchy and its Lax representation. First, we define a new vector field ∂_{τ_k} by a linear combination of all vector fields ∂_{t_n} in ordinary q -deformed KP hierarchy, then we introduce a new Lax type equation which consists of the τ_k -flow and the evolutions of wave functions. Under the evolutions of wave functions, the commutativity of ∂_{τ_k} -flow and ∂_{t_n} -flows gives rise to a new extended q -KP hierarchy. This new extended q -KP hierarchy contains two types of q -deformed KP equation with self-consistent sources (q -KPSCS-I and q -KPSCS-II), and its two kinds of reductions give the q -deformed Gelfand-Dickey hierarchy with self-consistent sources and the constrained q -deformed KP hierarchy, which are some $(1+1)$ -dimensional q -deformed soliton equation with self-consistent sources, e.g., two types of q -deformed KdV equation with self-consistent sources (q -KdVSCS-I and q -KdVSCS-II) and two types of q -deformed Boussinesq equation with self-consistent sources (q -BESCS-I and q -BESCS-II). The q -KdVSCS-II is just the q -deformed Yajima-Oikawa equation. All of these results reduce to the classical ones when $q \rightarrow 1$. Thus, the method proposed in this paper is a general way to find the $(1+1)$ - and $(2+1)$ -dimensional q -deformed soliton equation with self-consistent sources and their Lax representations.

The q -deformed differential operator ∂_q is defined as $\partial_q(f(x)) = \frac{f(qx) - f(x)}{x(q-1)}$, which recovers the ordinary differentiation $\partial_x(f(x))$ as $q \rightarrow 1$. Let us define the q -shift operator θ as $\theta(f(x)) = f(qx)$. For a q -pseudo-differential operator (q -PDO) of the form $P = \sum_{i=-\infty}^n p_i \partial_q^i$, we decompose P into the differential part and the integral part as follows $P_+ = \sum_{i \geq 0} p_i \partial_q^i$, $P_- = \sum_{i \leq -1} p_i \partial_q^i$. The conjugate operation “ $*$ ” for P is defined by $P^* = \sum_i (\partial_q^*)^i p_i$, $\partial_q^* = -\partial_q \theta^{-1} = -\frac{1}{q} \partial_{\frac{1}{q}}$.

The q -KP hierarchy is defined by the Lax equation

$$\partial_{t_n} L = [B_n, L], \quad B_n = L_+^n, \quad (1)$$

with Lax operator of the form $L = \partial_q + \sum_{i=0}^{\infty} u_i \partial_q^{-i}$.

For any fixed $k \in \mathbb{N}$, we define a new variable τ_k whose vector field is

$$\partial_{\tau_k} = \partial_{t_k} - \sum_{i=1}^N \sum_{s \geq 0} \zeta_i^{-s-1} \partial_{t_s},$$

where ζ_i 's are arbitrary distinct non-zero parameters, and introduce a new Lax type equation given by

$$L_{\tau_k} = [B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i, L]. \quad (2a)$$

with

$$\phi_{i,t_n} = B_n(\phi_i), \quad \psi_{i,t_n} = -B_n^*(\psi_i), \quad i = 1, \dots, N. \quad (2b)$$

Then we have the following proposition.

Proposition 1 (1) and (2) give rise to the following new extended q -deformed KP hierarchy

$$B_{n,\tau_k} - (B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i)_{t_n} + [B_n, B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i] = 0, \quad (3a)$$

$$\phi_{i,t_n} = B_n(\phi_i), \quad \psi_{i,t_n} = -B_n^*(\psi_i), \quad i = 1, \dots, N. \quad (3b)$$

Under (3b), the Lax representation for (3a) is given by

$$\Psi_{\tau_k} = (B_k + \sum_{i=1}^N \phi_i \partial_q^{-1} \psi_i)(\Psi), \quad \Psi_{t_n} = B_n(\Psi). \quad (4)$$

The two kinds of reductions of the new extended q -deformed KP hierarchy (3), the n -reduction and the k -constraint, give the q -deformed Gelfand-Dickey hierarchy with self-consistent sources and the constrained q -deformed KP hierarchy respectively. All of these results reduce to the classical ones when $q \rightarrow 1$.

References

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