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# Multiscale expansion on the lattice and integrability of dispersive partial difference equations

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Multiscale perturbation techniques have proved to be important tools for finding approximate solutions to many physical problems by reducing a given PDE to a simpler equation, which in many cases turn out to be integrable. These multiscale expansions are structurally strong and can be applied to both integrable and non-integrable systems. Zakharov and Kuznetsov in 1986 have shown that, starting from an integrable PDE and performing a proper multiscale expansion, one may obtain other integrable systems. In particular, they showed that the slow-varying amplitudes of a dispersive wave solution of the Nonlinear Schroedinger equation (NLS) satisfies the Korteweg-de Vries equation (KdV) and viceversa. Calogero et al. have used the multiscale perturbation technique as a tool to give necessary conditions for the integrability of large classes of PDE?s both in 1+1 and 2+1 dimensions. As a consequence of their theorem the non-integrability of the resulting multiscale reduction is a consequence of the non-integrability of the ancestor system. Later on, by extending the multiscale techniques to higher orders in the perturbation parameter, Degasperis, Manakov, Procesi and Santini have been able to prove the integrability of few new PDEs.

The extension of this approach to discrete equations have been proposed recently by many authors. One is mainly looking for a multiscale expansion technique on the lattice which, starting from dispersive integrable Z2-lattice equations, provides a reduced equation on a different rescaled Z2-lattice. To do so one had to introduce slow-varying conditions for the fields appearing in the discrete equation. The slow-varying conditions consist in requiring that the p+1 variation of the field be small, p being a positive integer. It has been proved that the resulting reduction of an integrable equation turns out to be non-integrable, contraddicting the basic intuitive ansatz introduced by Calogero, Eckhaus, Kuznetsov and Zhakarov. However, it has been recently shown that if p is infinite the reduced equations become formally continuous and the integrability property may be preserved by the reductive perturbation procedure. In this way the multiscale expansions easily works for both difference-difference and differential-difference equations.

In the present presentation we briefly review the results up to now obtained on the multiscale expansion of partial difference and differential-difference equations and we present, using many examples, results which suggest the possible extension to the discrete of all theorems so far obtained for partial differential equations. At the end we give a first classification theorem for multilinear dispersive discrete equations defined on a square.

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