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Orthogonal polynomials, random matrix theory and integrable systems.

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1 Orthogonal polynomials

The theory of orthogonal polynomials has seen many remarkable developments during the last two decades, due to its connections with integrable systems, spectral theory and random matrices. I will give an overview of some of these developments that were inspired by random matrix theory. Several other aspects will be treated in the other talks in the miniworkshop.

The story starts with the formulation of a Riemann-Hilbert problem for orthogonal polynomials [8] which was later used for the complete asymptotic analysis by means of the Deift/Zhou steepest descent analysis [5], [6]. Along the way many important issues in random matrix theory where settled including universality of local spacing distributions, large gap asymptotics and phase transitions.

2 Multiple orthogonal polynomials

I will also include a discussion of the recently developed connection of certain random matrix models with multiple orthogonal polynomials [3]. These are generalizations of usual orthogonal polynomials where the orthogonality is distributed over two or more orthogonality weights. These polynomials have an integrable structure that is related to multi component KP equations [1].

Several models from random matrix theory and related areas have been treated with multiple orthogonal polynomials. These models include Gaussian random matrices with external source and non-intersecting Brownian paths [4], non-intersecting squared Bessel paths [9], coupled random matrices with a quartic potential [7], and the Cauchy matrix model [2].

All these models give rise to larger size Riemann-Hilbert problems (larger than 2×2) and have new critical phenomena. The asymptotic analysis of the larger size Riemann-Hilbert problems presents a number of surprising new features, and we are now only seeing the beginning of it. One aspect is the global eigenvalue distribution which is described by vector equilibrium problems. Another aspect is the appearance of new kinds of phase transitions that cannot occur in the usual random matrix models. Here much work remains to be done.

References

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