

# Combinatoric and geometry in shallow water: Soliton solutions of the KP equation

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March 23, 2009

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## 1 Introduction

The KP equation describes two-dimensional shallow water waves, which is an two-dimensional extension of the KdV equation,

$$\frac{\partial}{\partial x} \left( -4 \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) + 3 \frac{\partial^2 u}{\partial y^2} = 0.$$

I will explain a classification of the soliton solutions of the KP equation based on the Schubert decomposition of the Grassmannian  $\text{Gr}(N, M)$  [5, 1, 2, 3]. Each soliton solution is then identified as a point on a *totally nonnegative* Grassmann cell. The simplest case is one line-soliton (just like a beach wave), and it is identified as a point on  $\text{Gr}(1, 2)$ . Also if the solution contains asymptotically *two* line-solitons for both  $y \rightarrow \pm\infty$ , then there are only *seven* distinct solutions (four line-solitons could be all different). Those solutions are parametrized by the *derangements* with just *two* excedances in the permutation group  $S_4$ , which correspond to the points on  $\text{Gr}(2, 4)$ . Furthermore, some of those exact solutions can be used to describe two dimensional wave patterns observed in shallow water experiments, which are generated by the resonant interactions of soliton solutions related to the Mach reflections [4].

## 2 Background and Summary

The soliton solutions of the KP equations considered in this paper are expressed in the form  $u = 2(\ln \tau)_{xx}$  where the  $\tau$ -functions are given by the Wronskian determinant of  $N$  independent functions  $\{f_1, f_2, \dots, f_N\}$ ,

$$\tau(x, y, z) = \text{Wr}(f_1, f_2, \dots, f_N), \quad (1)$$

where  $f_i$ 's satisfy the linear equations,

$$\frac{\partial f_i}{\partial y} = \frac{\partial^2 f_i}{\partial x^2}, \quad \frac{\partial f_i}{\partial t} = \frac{\partial^3 f_i}{\partial x^3}.$$

Considering a set of finite dimensional solutions  $\{f_1, \dots, f_N\}$  in the form,

$$f_i = \sum_{j=1}^M a_{i,j} E_j \quad \text{with} \quad E_j = \exp(k_j x + k_j^2 y + k_j^3 t),$$

with  $k_1 < k_2 < \dots < k_M$ , the  $\tau$ -function can be identified as a point of  $\text{Gr}(N, M)$ , and the  $A$ -matrix defined by  $A = (a_{i,j})$  give a parametrization of the point on the Schubert cell of  $\text{Gr}(N, M)$ . For a physically interesting solution, we look for a regular solution, which corresponds to a point on a *totally nonnegative* Grassmann cell obtained by a further refinement of the Schubert decomposition. For example, one line-soliton is given by  $\tau = a_{11}E_1 + a_{12}E_2$  with  $a_{11}a_{12} > 0$ , that is, there are two 2-cells with  $a_{11}a_{12} = \pm$ , and the regular solution is on the positive cell with  $a_{11}a_{12} > 0$ . We denote one line-soliton as  $[1, 2]$ -soliton. This indicates that  $E_1$  dominates in  $x \ll 0$  and as increasing  $x$ ,  $E_2$  becomes dominant over  $E_1$ , i.e. exchanging the indices  $1 \leftrightarrow 2$  (*permutation*). In general we have shown the classification theorem [1, 2, 3]:

**Theorem 1** *Let  $A$  be an irreducible  $N \times M$  matrix in the reduced row echelon form with the pivot indices given by  $\{e_1, \dots, e_N\}$  and the non-pivot by  $\{g_1, \dots, g_{M-N}\}$ . Then the solution generated by the  $\tau$ -function (1) with the  $A$ -matrix has asymptotically*

- $N$  line-solitons of  $[e_n, j_n]$ -types with  $e_n < j_n$  for  $y \gg 0$ ,
- $(M - N)$  line-solitons of  $[i_m, g_m]$ -types with  $i_m < g_m$  for  $y \ll 0$ .

Here we have  $\{1, 2, \dots, M\} = \{j_1, \dots, j_N, i_1, \dots, i_{M-N}\}$ , that is, the map defined by  $\pi(e_n) = j_n$  and  $\pi(g_m) = i_m$  is a bijection on the permutation group  $S_N$ , and the irreducibility of  $A$ -matrix means that the corresponding  $\tau$ -function cannot be identified as a point on a smaller Grassmannian  $\text{Gr}(N', M')$  with  $N' \leq N$  or  $M' \leq M$ .

The classification problem of all regular solutions with our  $\tau$ -function then leads to the parametrization of totally non-negative Grassmann cells, and each regular solution can be expressed by a unique (open) *chord diagram* [1].

We also consider *deformations* of the chord diagrams to study the initial value problems of the KP equation whose initial data are somewhat close to the exact solutions. In the presentation, I will show several pictures and movies based on direct simulations of the KP equation and *real* experiments of shallow water waves.

## References

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