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Autoresonant asymptotics in the oscillating system with weak dissipation

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Statement of the problem. The purpose of the contribution consists in analysis of the autoresonance phenomenon occured in nonlinear oscillating systems with dissipation. A system of the main resonance equations is studied

$$\frac{d\rho}{d\theta} = f(\theta)\sin\psi - \beta\rho, \quad \rho\Big[\frac{d\psi}{d\theta} + \lambda\theta - \rho^2\Big] = g(\theta)\cos\psi; \quad (\beta, \lambda = \text{const} > 0). \tag{1}$$

Dissipation corresponds to the positive factor β. Unlimited growth at infinity of the time depending factor $f(\theta) = f_0 + f_1\theta,\ f_1 \neq 0;\quad g(\theta) = g_0 + g_1\theta$ is a specificity of the problem. The main mathematical achievements are connected with construction of an asymptotics at infinity for two-parameter solution with growing amplitude $\rho(\theta) \approx \sqrt{\lambda\theta},\ \theta \to \infty$. Such solutions describe an initial stage of the autoresonance capture phenomenon.

Origin of the problem. Autoresonance is a phase locking phenomenon occurring in nonlinear oscillatory system, which is forced by oscillating perturbation with slow varying frequency. The essence of the phenomenon is that the nonlinear oscillator selfadjusts to the varying external conditions so that it remains in resonance with the driver for a long time. This long time resonance leads to a strong increase in the response amplitude under weak driving perturbation. Many applications of the autoresonance are known in nonlinear physics [1].

As an example of the original mathematical model one can considered a nonlinear oscillator:

$$\frac{d^2x}{dt^2} + U'(x) = -\beta_0\frac{dx}{dt} + \varepsilon F(t)\cos(\phi(t)). \tag{2}$$

Here the potential U is a smooth function with asymptotics $U(x) = x^2/2 + \gamma x^4/4 + \mathcal{O}(x^6),\ x \to 0$ so that the point $x = 0$ is a stable equilibrium of the unperturbed equation. The right hand side represents a perturbation with small parameters $0 < \beta_0, \varepsilon \ll 1$. It is supposed that both the driver frequency $\dot\phi(t) = \omega + \alpha t$ and the driver amplitude $F = f_0 + \delta t$ are slowly varying; it means $|\alpha|, |\delta| \ll 1$. Such equations with nonzero dissipation are occurred in theory of magnetodynamics for a thin ferromagnetic film in a weak external magnetic field. Solutions with an initial data near equilibrium $|x(0)|, |\dot x(0)| \ll 1$ are investigated.

The basic feature of the problem is the resonant condition $\dot\phi(0 = 1$. It means that the driver frequency in the initial moment coincides with the free frequency of linearized oscillator. The autoresonance corresponds to solutions of the perturbed equations which amplitudes increase up to order of unity while a perturbation remains small. Capture in such resonance take place not always, it depends on the initial data. In order to identify the cases of resonance capture we have to considered the equations on an initial stage where the times are not too long. Just in this approach the main resonance equations (1) arise. They appear in averaging original equations with small perturbation. The functions $\rho, \psi(\theta)$ represent slowly varying amplitude and phase shift of the fast harmonious oscillations near equilibrium:

$$x = \varepsilon^{1/3}\frac{2\,\mathrm{sgn}\gamma}{\sqrt{3|\gamma|}}\rho(\theta)\cos(\phi+\psi(\theta)) + \mathcal{O}(\varepsilon^{2/3}), \quad \theta = \varepsilon^{2/3}t. \tag{3}$$

Then the factors in the equations (1) are calculated by the formulas: $\beta = \beta_0/2\varepsilon,\ \lambda = \alpha/\varepsilon^{4/3}\mathrm{sgn}\gamma,\ f_1 = g_1 = -\delta\sqrt{3|\gamma|}/4\varepsilon^{2/3},\ f_0 = g_0 = -F_0\sqrt{3|\gamma|}/4$.

Results. It is not supposed any small parameters in the average equations (1); for their solutions an asymptotics at infinity as $\theta \to \infty$ is investigated. The autoresonance is identified with existence of solutions which have an unlimited growing amplitude $\rho(\theta)$. A problem of such type was earlier studied at absence of the dissipation and under a constant pumping amplitude. In real physical systems, in particular, in magnetodynamic systems there are dissipative effect which can be small and nevertheless one leads to damping of the free oscillations. In the case of a small amplitude pumping the dissipation can effect full suppression of the resonant effect. In particular, the autoresonance is not possible in a system of type (2) under both a constant pumping amplitude and $\beta_0 > 0$. In order to compensate dissipation losses of energy and to keep system in the autoresonant mode we offer to increase slowly the pumping amplitude: $F = f_0 + f_1\theta$. Just on this way the equations (1) with $f_1 \neq 0$ occur and there are suitable solutions with a growing amplitude $\rho(\theta) \approx \sqrt{\lambda\theta},\ \theta \to \infty$. One has to note, that the leading order term of asymptotics with growing amplitude $\rho(\theta)$ is not connected with growth of pumping amplitude $f_1\theta$. The pumping amplitude despite of it growth remains small on the times $t \approx \alpha^{-1}$ which are specific for the autoresonance at $\alpha = \mathcal{O}(\varepsilon^{4/3})$. Growing of the energy up to order of unity is primarily obliged to the resonance.

Theorem. Let be $\lambda > 0$ and $(1 + f_1/g_1)\beta > 0$. Then the equations (1) have a two-parametrical family of solutions which have asymptotics

$$\rho(\theta) = \sqrt{\lambda\theta}[1 + o(\theta^{-1/4})], \quad \psi(\theta) = o(1), \quad \theta \to \infty. \tag{4}$$

References

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