NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems
Isola Rossa, May 16–23, 2009.

Characteristic Lie algebras and the classification of integrable discrete chains

Ismagil Habibullin[a], Natalya Zheltukhina[b], and Aslı Pekcan[b]

February 26, 2009

a. Ufa Institute of Mathematics, Russian Academy of Science, Chernyshevskii Str., 112, Ufa, 450077, Russia.

b. Department of Mathematics, Faculty of Science, Bilkent University, 06800, Ankara, Turkey.

1 Darboux integrable chains

We study differential-difference equation of the form

$$t_x(n+1) = f(t(n), t(n+1), t_x(n)), \tag{1}$$

with unknown $t = t(n, x)$ depending on x, n. Equation (1) is called Darboux integrable, if two functions of a finite number of arguments $F(x, t(n), t(n\pm1), t(n\pm 2), ...)$ and $I(x, t(n), t_x(n), t_{xx}(n), ...)$ called x- and n-integrals correspondingly exist, such that $D_xF = 0$ and $DI = I$, where D_x is the operator of total differentiation with respect to x, and D is the shift operator: $Dp(n) = p(n+1)$. The Darboux integrability property can be reformulated in terms of characteristic Lie algebras that gives an effective tool for classification of integrable equations.

2 Algebraic criterion of integrability

To introduce the characteristic Lie algebra L_n of (1) in the direction of n, note that $D^{-j}\frac{\partial}{\partial t_1}D^jI = 0$ for any n-integral I and $j \geq 1$, where $t_k = t(n+k)$. Indeed, the equation $DI = I$ can be rewritten in an enlarged form as

$$I(x, n+1, t_1, f, f_x, f_{xx}, \ldots) = I(x, n, t, t_x, t_{xx}, \ldots)\,. \tag{2}$$

The left hand side DI of equality (2) contains the variable t_1, while the right hand side does not. Hence, $\frac{\partial}{\partial t_1}(DI) = 0$ that implies $D^{-1}\frac{\partial}{\partial t_1}DI = 0$. Proceeding this way one can easily prove the statement above from the equality $D^jI = I$, $j \geq 1$.

Define vector fields $Y_j = D^{-j}\frac{\partial}{\partial t_1}D^j$ and $X_j = \frac{\partial}{\partial t_{-j}}$ for $j \geq 1$. We have, $Y_jI = 0$ and $X_jI = 0$ for any n-integral I of (1) and $j \geq 1$. The following theorem defines the characteristic Lie algebra L_n of (1).

Theorem 1. Equation (1) admits a nontrivial n-integral if and only if the following two conditions hold:

1) Linear space spanned by the operators $\{Y_j\}_1^\infty$ is of finite dimension, denote this dimension by N;
2) Lie algebra L_n generated by the operators $Y_1, Y_2, ..., Y_N, X_1, X_2, ..., X_N$ is of finite dimension. We call L_n the characteristic Lie algebra of (1) in the direction of n.

To introduce the characteristic Lie algebra L_x of (1) in the direction of x, rewrite the equation (1) in the "inverse" form as $t_x(n-1) = g(t(n), t(n-1), t_x(n))$. An x-integral $F(x, t, t_{\pm 1}, t_{\pm 2}, \ldots)$ solves the equation $D_x F = 0$, i.e. $K_0 F = 0$, where $f_1 = Df$, $g_{-1} = D^{-1}g$, etc.

$$K_0 = \frac{\partial}{\partial x} + t_x \frac{\partial}{\partial t} + f \frac{\partial}{\partial t_1} + g \frac{\partial}{\partial t_{-1}} + f_1 \frac{\partial}{\partial t_2} + g_{-1} \frac{\partial}{\partial t_{-2}} + \ldots . \tag{3}$$

Since F does not depend on the variable t_x one gets $XF = 0$, where $X = \frac{\partial}{\partial t_x}$. Therefore, any vector field from the Lie algebra generated by K_0 and X annulates F. This algebra is called the characteristic Lie algebra L_x of the chain (1) in the x-direction.

The following result is essential, its proof is a simple consequence of the famous Jacobi theorem (Jacobi theorem is discussed, for instance, in [1] where the notion of the characteristic Lie algebra was introduced for the hyperbolic type partial differential equations).

Theorem 2. Equation (1) admits a nontrivial x-integral if and only if its Lie algebra L_x is of finite dimension.

Some classification results (see [2], [3]) will be presented which have been obtained by using characteristic Lie algebras. For example:

Theorem 3. Chain $t_{1x} = t_x + d(t, t_1)$ admits a nontrivial x-integral if and only if $d(t, t_1)$ is one of the kind:

(1) $d(t, t_1) = c_1 t(t - t_1) + c_2(t - t_1)^2 + c_3 t - c_3 t_1,$

(2) $d(t, t_1) = A(t - t_1)\mathrm{e}^{\alpha t},$

(3) $d(t, t_1) = c_4(\mathrm{e}^{\alpha t_1} - \mathrm{e}^{\alpha t}) + c_5(\mathrm{e}^{-\alpha t_1} - \mathrm{e}^{-\alpha t}),$

where $A = A(\tau)$ is an arbitrary function of $\tau = t - t_1$ and $\alpha, c_1, ...c_5$ are some constants.

References

[1] A. N. Leznov, V. G. Smirnov, A. B. Shabat, *Group of inner symmetries and integrability conditions for two-dimensional dynamical systems,* Teoret. Mat. Fizika, **51**, no. 1, 10-21 (1982).

[2] Ismagil Habibullin, Natalya Zheltukhina, Aslı Pekcan, *On the classification of Darboux integrable chains*, JMP **49**, 102702 (2008) //arXiv : nlin/0806.3144.

[3] I. Habibullin, A. Pekcan, *Characteristic Lie Algebra and Classification of Semi-Discrete Models*, Theoret. and Math. Phys., **151**, no. 3, 781-790 (2007) //arXiv : nlin/0610074.