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Structure of Noether's Conservation Laws

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1 Extended Abstract

Noether's Theorem provides a conservation law given that a Lagrangian is invariant under a group action. We will show that Noether's conservation laws can also be written as the product of a representation of a right moving frame, which is equivariant, and a matrix of invariants. (For moving frames see [1].)

For example, the conservation laws coming from the invariant Lagrangian $\kappa^2 \mathrm{d}s$ under the Euclidean action can be written as

$$\begin{pmatrix} \frac{1}{\sqrt{1+u_x^2}} & \frac{-u_x}{\sqrt{1+u_x^2}} & 0 \\ \frac{u_x}{\sqrt{1+u_x^2}} & \frac{1}{\sqrt{1+u_x^2}} & 0 \\ \frac{xu_x - u}{\sqrt{1+u_x^2}} & \frac{uu_x + x}{\sqrt{1+u_x^2}} & 1 \end{pmatrix} \begin{pmatrix} -\kappa^2 \\ -2\kappa_s \\ 2\kappa \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

where the first conservation law comes from the translation in x, the second comes from the translation in u, and the third conservation law comes from the rotation in the (x, u)-plane.

It was already known from [3] that there is a group action on the conservation laws explicitly. So here we will show that the adjoint action on a vector field generating the group yields the correct representation of the right moving frame. Also we will show that the matrix of invariants comes partly from the derivation of the Euler-Lagrange equations. The form used here to obtain the Euler-Lagrange equations completes the Kogan-Olver [2] way of obtaining them. The result is summarised in the following theorem.

Noether's Theorem via Moving Frames (Gonçalves, Mansfield)
Let $\int L(\kappa^\alpha, \kappa^\alpha_x, ...)\mathrm{d}x$ be invariant under $G \times M \to M$, where $M = J^N((x, u^\alpha))$, with generating invariants κ^α and $g \cdot x = x$.

Introduce a dummy variable τ to effect de variation and suppose that

$$\frac{\partial}{\partial \tau} \int L \mathrm{d}x = \int \sum_\alpha \mathsf{E}^\alpha(L) I_2^\alpha + \frac{\mathrm{d}}{\mathrm{d}x} \left[\sum_{\alpha, J=1\cdots 1} I^\alpha_{2J} C^\alpha_J \right],$$

where this defines the vector $\mathcal{C}^\alpha = (C^\alpha_J)$. Let $(a_1, ..., a_p)$ be the coordinates of G near the identity e. Define the matrix of invariantized infinitesimals

$$\Omega^\alpha = \begin{array}{c} \\ \vdots \\ a_j \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} \cdots & \quad u^\alpha_J \quad & \cdots \end{array} \\ \left(\begin{array}{ccc} \cdots & \cdots & \cdots \\ \cdots & \left(\left. \dfrac{\partial (g \cdot u^\alpha_J)}{\partial a_j} \right|_e \right) \Big|_{\text{frame}} & \cdots \\ \cdots & \cdots & \cdots \end{array} \right) \end{array}.$$

Then the p conservation laws obtained via Noether's Theorem can be written in the form

$$Ad(\rho)^{-1} \sum_\alpha \Omega^\alpha \mathcal{C}^\alpha = \mathbf{c},$$

where $Ad(\rho)$ is the representation of the moving frame ρ.

This result is a consequence of our method. Thus, Noether's Theorem calculates the moving frame for historically known invariants, giving us the frame even when it is not possible to solve for it.

References

[1] Mark Fels, and Peter J.Olver, *Moving Coframes I*, Acta Appl. Math. 51 (1998) 161-312.

[2] Irina A. Kogan, and Peter J. Olver, *Invariant Euler-Lagrange Equations and the Invariant Variational Bicomplex*, Acta Appl. Math. 76 (2003) 137-193.

[3] Peter J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Springer Verlag, New York, 1993.