NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems
Isola Rossa, May 16–23, 2009.

An extended class of orthogonal polynomials defined by a Sturm-Liouville problem

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March 1, 2009

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1 Introduction

The classical orthogonal polynomial systems (OPS) of Hermite, Laguerre and Jacobi are most often characterized as the polynomial solutions of a Sturm-Liouville problem. Following the celebrated result by S. Bochner: if an infinite sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfies a second order eigenvalue equation of the form

$$p(x)P_n'' + q(x)P_n' + r(x)P_n(x) = \lambda_n P_n(x), \qquad n = 0, 1, 2, \dots \tag{1}$$

then $p(x), q(x)$ and $r(x)$ must be polynomials of degree $2, 1$ and 0 respectively [7, 2]. In addition, if the $\{P_n(x)\}_{n=0}^{\infty}$ sequence is an OPS, then it has to be (up to an affine transformation of x) one of the classical orthogonal polynomial systems of Jacobi, Laguerre or Hermite

It seems to be a well established fact in the literature that no complete orthogonal polynomial systems other than the classical ones arise as solutions of a Sturm-Liouville problem. This is indeed the case if the operator belongs to the Bochner class (1), as was proved by Lesky [5]. However, we argue that from the point of view of Sturm-Liouville theory this restriction is not essential. It has been observed [3] that certain instances of classical orthogonal polynomial families have the following curious property: the polynomials are formal eigenfunctions of the operator (1), but a finite number of initial polynomials are not square integrable. Consider, for instance the family of Laguerre polynomials $P_n(x) = L_n^{-1}(x)$, $n = 0, 1, 2, \dots$. Since the orthogonality is with respect to the weight $W(x)dx = x^{-1}e^{-x}dx$, $P_0(x)$ is not square integrable; only the polynomials $P_1, P_2, P_3, \dots$ arise as eigenfunctions of the corresponding Sturm-Liouville problem.

The following question is therefore of interest:

> *What sequences of polynomials can arise as eigenfunctions of a Sturm-Liouville problem?*

The main idea of our presentation is to show that the answer to the above question takes one outside the realm of classical orthogonal polynomials. The polynomials we present below feature rational modification of classical weights. However they cannot be obtained through the use of Uvarov's formula[8, 4]; the polynomials presented below are not semi-classical[1, 6].

2 Summary of results

Definition 2.1 *We define a* polynomial Sturm Liouville problem *(PSLP) to be a self-adjoint Sturm-Liouville boundary value problem with a semi-bounded, pure-point spectrum and* polynomial eigenfunctions*. For integer $k \geq 0$, we will say that a polynomial sequence $\{y_n\}_{n=k}^{\infty}$ is a k-OPS and if it is a complete, orthogonal basis relative to a positive measure $W(x)\,dx$*

We are interested in in PSLPs outside the Bochner class whose eigenfunctions form a k-OPS for $k \geq 1$. Let $\alpha \neq \beta$ be real parameters and let

$$a = \frac{1}{2}(\beta - \alpha), \quad b = \frac{\beta+\alpha}{\beta-\alpha}, \quad c = b + 1/a. \tag{2}$$

We define the X1 Jacobi polynomials to be the orthogonalization of the polynomial sequence

$$x - c, \quad (x-b)^i, \; i = 2, 3, \ldots$$

relative to the weight

$$W(x) = \frac{(1-x)^\alpha (1-x)^\beta}{(x-b)^2}, \quad -1 < x < 1.$$

We define the X1 Jacobi SLP to be the boundary value problem

$$T(y) = (x^2-1)y'' + 2a\left(\frac{1-b\,x}{b-x}\right)\big((x-c)y' - y\big), \tag{3}$$

subject to the boundary conditions

$$\lim_{x\to 1^-} (1-x)^{\alpha+1}(y(x) - (x-c)y'(x)) = 0, \tag{4}$$

$$\lim_{x\to -1^+} (1+x)^{\beta+1}(y(x) - (x-c)y'(x)) = 0. \tag{5}$$

Our main result is the following.

Theorem 2.1 *The X_1-Jacobi and the similarly defined X_1-Laguerre boundary value problems are PSLPs. Their respective eigenfunctions are the X_1-Jacobi and X_1-Laguerre 1-OPSs. Conversely, if all the eigenpolynomials of a PSLP form a 1-OPS, then up to an affine transformation of the independent variable, the family in question is either a classical 1-OPS, or X_1-Jacobi, or X_1-Laguerre.*

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