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Some exact solutions for a generalized Ostrovsky equation

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1 Abstract

To describe long internal waves in a rotating ocean, Ostrovsky [2] derived the approximate nonlinear equation

$$(u_t + (u^2)_x - \beta u_{xxx})_x = \gamma u \tag{1}$$

with $x \in \mathbb{R}$ and $\gamma > 0$, where β and γ are dispersion coefficients. For the Ostrovsky equation certain approximate stationary solutions [3] have been constructed (no accurate solutions are known), and the dynamics of individual non-stationary perturbations of the soliton type have been studied. The Ostrovsky equation is a model for weakly nonlinear long internal and surface waves in a rotating ocean. It was shown by Leonolov and Galkin in [4, 5] that Eq. (1) does not have steady solitary wave solutions. In view of the complication of the unidirectional propagation in rotating fluid, the changing rate for the dispersive term and convection term are usually proportional to a polynomial of velocity. In a recent paper Tian and Yin introduced the generalized Ostrovsky equation (GOE) with a polynomial of velocity as follows

$$(u_t + (u^2)_x - \beta u_{xxx})_x = \gamma f(u), \tag{2}$$

where $f(u)$ is a polynomial of velocity, namely $f(u) = \sum_{i=1}^{n} k_i u^i$, n being a constant. When $f(u) = u$, (2) is the Ostrovsky equation. In [1] the authors found that when the polynomial of velocity is quadratic i.e. $f(u) = u^2 + u$ the generalized Ostrovsky equation has new solitary wave solutions and Backlund transformations. In this work, we study equation (2) from the point of view of the theory of symmetry reductions in partial differential equations. We obtain the classical symmetries admitted by (2) for arbitrary f and the functional forms of f for which equation (2) admits extra classical symmetries [6], then, we use the transformations groups to reduce the equations to ordinary differential equations. In [7] Kudryashov pointed out that many of the so called "new travelling wave solutions" could be derived from the solutions of the simple nonlinear ordinary differential equation

$$(h_z)^2 + 2(h-a)(h-b)(h-c) = 0, \tag{3}$$

where a, b and c are the roots of the algebraic equation

$$h^3 - \frac{1}{2}\lambda h^2 + c_1 h + c_2 = 0. \tag{4}$$

Equation (3) has the solutions

$$h = (a-b)\mathrm{cn}^2\left(\sqrt{\frac{a-c}{2}}z, \frac{b-a}{c-a}\right) + b, \tag{5}$$

$$h = (b-a)\mathrm{sn}^2\left(\sqrt{\frac{a-c}{2}}z, \frac{b-a}{c-a}\right) + a. \tag{6}$$

When (4) has two equal roots we get solutions for the ordinary differential equation that lead to solitary waves of the corresponding partial differential equation.

Due to the fact that equation (2) admits extra symmetries if and only if $f(u)$ is a cubic polynomial $f(u) = (au+c)^3$ we look for travelling wave solutions of the generalized Ostrovsky equation when $f(u)$ is a cubic polynomial that is

$$(u_t + (u^2)_x - \beta u_{xxx})_x = \gamma(k_1 u^3 + k_2 u^2 + k_3 u + k_4). \tag{7}$$

To find travelling wave solutions we take $u(x,t) = h(z)$ where $z = x - \lambda t$ and we get the reduced ordinary differential equation

$$-\beta h'''' + 2hh'' - \lambda h'' + 2(h')^2 = k_1 h^3 + k_2 h^2 + k_3 h + k_4. \tag{8}$$

We obtain the following: Equation (8) admits solutions (5) and (6) of equation (3) if k_i $i = 1, \ldots 4$, a, b, c, β and λ satisfy four conditions. Consequently (2) has many exact solutions that can be expressed in terms of the Jacobi elliptic functions. Hence the GOE equation has many exact solutions that can be expressed in terms of trigonometric and hyperbolic functions and (2) has plenty periodic waves, solitary waves, compactons, etc.

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