NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems
Isola Rossa, May 16–23, 2009.

Limit-periodic sets and isochronous systems

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February 24, 2009

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1 Limit-periodic sets

Limit-periodic sets have been introduced [2] to study polynomial planar vector fields in relation with Hilbert's 16th problem about limit cycles. They can be defined in full generality for any topological dynamical system as follows. Let E be a metric space. We consider a family of topological Dynamical Systems as a family of continuous maps depending continuously of a parameter $\lambda \in D$, where D is a metric space:

$f_\lambda : R \times E \to E$, $f_\lambda : (t,p) \mapsto f_\lambda(t,p)$ such that

(i) $f_\lambda(0,p) = p$

(ii) $f_\lambda(t_1, f_\lambda(t_2,p)) = f_\lambda(t_1+t_2,p)$.

(iii) For all p, and t fixed,

$$\lambda \mapsto f_\lambda(t,p)$$

is a continuous map from D to E.

Variable t is called time. Set of points

$$\{f_\lambda(t,p), t \in R\},$$

is the trajectory or the solution with initial data p. Set

$$\{f_\lambda(t,p), t \in R_+\},$$

respectively

$$\{f_\lambda(t,p), t \in R_-\},$$

is the positive half-trajectory of p, respectively the negative half-trajectory. Set

$$\{f(t,p), T_1 < t < T_2,\}$$

is an arc of the trajectory of p of duration $T_2 - T_1$.

An orbit of a dynamical system is called periodic of period T if it is not a fixed point and if for all $t \in \mathbb{R}$, $f(t+T,p) = f(t,p)$. The set of such T is a $\mathbb{Z}$-module called the period module. The minimal period is the smallest positive element of the period module. We call the associated set $\gamma = \{f(t,p), 0 \le t < T\}$ the periodic orbit. A periodic orbit is a compact connected invariant set for the dynamical system.

Recall that if E, d is a metric space the set $K(E)$ of all compact sets of E is a metric space for the Hausdorff distance:

$$d_H(A, B) = \mathrm{Sup}_{x \in A, y \in B}\{\mathrm{Inf}_{z \in B} d(x, z), \mathrm{Inf}_{z' \in A} d(z', y)\}.$$

The topology defined on $K(E)$ is independent of d. If E, d is a compact metric space then $K(E)$ is compact.

Definition

A limit periodic set Γ of a topological dynamical system $f_0 : (t, p) \mapsto f_0(t, p)$ defined on E is a compact set in E such that there exists a convergent sequence $\lambda_n \to 0$ in the parameter space D, and a sequence of periodic orbits γ_n of f_{λ_n} which converges for the Hausdorff distance to Γ.

It seems quite natural to investigate these sets in relation with bifurcations of isochronous systems which have been uncovered recently by Francesco Calogero and his co-workers [1].

In this talk we first explain that the situation with two-dimensional dynamics is surprisingly already quite rich.

2 Limit-periodic sets of planar vector fields

Let m be a regular point of a limit-periodic set, there is a section of the flow which intersects the limit-periodic set in no other point than m. Hence limit-periodic sets are completely described by the theorem of Poincaré-Bendixson. But the theory of bifurcations of isochronous system shows that there are examples of non-isolated limit-periodic sets (which fill a set of full measure).

3 Limit-periodic sets and Birkhoff's recurrence theory

We show next that if the limit of the periodic sets is uniform, Birkhoff's theory about usual ω-limit sets applies and in that case, the limit-periodic set contains an almost-periodic solution.

References

[1] F. Calogero: Isochronous Systems Oxford University Press 2008

[2] JP Francoise, CC Pugh *Keeping track of limit cycles*, J. Differential Equations **65** (1986), p. 139-157.