NEEDS 2009 Workshop
Nonlinear Evolution Equations and Dynamical Systems
Isola Rossa, May 16–23, 2009.

On the spaces associated with the KP-equation

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February 27, 2009

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1 The Ricci flat 8-dim metric and KP-equation

In the papers [1]-[2] was showed that the eight-dimensional metric in local coordinates (x, y, z, t, P, Q, U, V)

$$^{8}ds^2 = 2\left(-PH_{11} - P\frac{\partial F}{\partial t} - H_{12}Q - 2\Gamma^3_{11}U + H_{22}V\right)dx^2 +$$

$$+4\left(H_{11}Q - H_{12}V\right)dxdy + 4\left(-\frac{\partial F}{\partial z}V + \frac{\partial F}{\partial t}U\right)dxdz + 2(H_{11}U - H_{31}V)dy^2 +$$

$$+2dxdP + 2dydQ + 2dzdU + 2dtdV, \quad (1)$$

is the Ricci-flat $R_{ik} = 0$ if the conditions on the functions $H_{ij} = H_{ij}(y, z, t)$, $\Gamma^3_{11}(y, z, t)$ and $F(y, z, t)$

$$\frac{\partial H_{12}}{\partial y} - \frac{\partial H_{22}}{\partial t} = 0, \quad -\frac{\partial H_{11}}{\partial y} + \frac{\partial H_{21}}{\partial t} = 0, \quad -\frac{\partial H_{11}}{\partial z} + \frac{\partial H_{31}}{\partial t} = 0, \quad (2)$$

$$\frac{\partial \Gamma^3_{11}}{\partial z} = 2\left(\frac{\partial F}{\partial t}\right)^2 + 2H_{11}\frac{\partial F}{\partial t} + 2(H_{11})^2. \quad (3)$$

are hold.

The metric (1) has the form

$$ds^2 = -\Gamma^i_{jk}\xi_i dx^j dx^k + 2d\xi_k dx^k$$

and it is an example of Riemann extension of affinely-connected the four dimensional space in local coordinates x^l with symmetric connection $\Gamma^i_{jk}(x^l) = \Gamma^i_{kj}(x^l)$. From the system (2) after the substitution [3]

$$H_{11} = -\frac{1}{2}u(y, z, t), \quad H_{12} = -\frac{1}{3}v(y, z, t), \quad H_{21} = -\frac{2}{3}v(y, z, t) - \frac{1}{2}\frac{\partial u(y, z, t)}{\partial t}$$

$$H_{31} = -\frac{3}{4}w(y, z, t) + \frac{3}{8}u(y, z, t)^2 - \frac{\partial v(y, z, t)}{\partial t} - \frac{1}{2}\frac{\partial^2 u(y, z, t)}{\partial t^2},$$

$$H_{22} = -\frac{1}{2}w(y, z, t) + \frac{1}{2}u(y, z, t)^2 - \frac{\partial v(y, z, t)}{\partial t} - \frac{1}{2}\frac{\partial^2 u(y, z, t)}{\partial t^2} + \frac{\partial u(y, z, t)}{\partial y},$$

the famous KP-equation follows

$$\frac{\partial}{\partial t}\left(\frac{\partial u(y, z, t)}{\partial z} - \frac{3}{2}u(y, z, t)\frac{\partial u(y, z, t)}{\partial t} - \frac{1}{4}\frac{\partial^3 u(y, z, t)}{\partial t^3}\right) = \frac{3}{4}\frac{\partial^2 u(y, z, t)}{\partial y^2}. \quad (4)$$

2 Four dimensional Ricci-flat affinely connected subspace and KP-equation

The main result of [2] is the following

Theorem The four-dimensional affinely connected space with non zero coefficients of connection $\Gamma^i_{jk} = \Gamma^i_{kj}$ of the form

$$\Gamma^1_{11} = H_{11} + \frac{\partial F}{\partial t}, \quad \Gamma^2_{11} = H_{12}, \quad \Gamma^3_{11} = \Gamma^3_{11},, \quad \Gamma^2_{12} = H_{11}, \quad \Gamma^3_{13} = -\frac{\partial F}{\partial t}$$

$$\Gamma^3_{22} = -H_{11}, \quad \Gamma^4_{11} = -H_{22}, \quad \Gamma^4_{12} = H_{21}, \quad \Gamma^4_{13} = \frac{\partial F}{\partial z}, \quad \Gamma^4_{22} = H_{31}$$

is a Ricci flat

$$R_{ij} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{im}\Gamma^m_{kj} = 0,$$

if the conditions (2-3) hold. The problem of metrizibility such type of connection is important in theory of 3-dim manifolds.

The following proposition may be useful in applications

Theorem The Ricci tensor of six-dimensional Plebanski space in local coordinates (x, y, z, u, v, v) with the metric of the form

$$^6ds^2 = A(\vec{x})du^2 + 2A(\vec{x})dudv + 2E(\vec{x})dudw + C(\vec{x})dv^2 + 2H(\vec{x})dvdw + F(\vec{x})dw^2 +$$
$$+ dxdu + dydv + dzdw$$

where all components of the metric depend on variables $\vec{x} = (x, y, z, u, v, w)$ has fefteen components.

Nine of them are equal to zero due the conditions [4]

$$\frac{\partial E(\vec{x})}{\partial x} + \frac{\partial H(\vec{x})}{\partial y} + \frac{\partial F(\vec{x})}{\partial z} = 0, \quad \frac{\partial B(\vec{x})}{\partial x} + \frac{\partial C(\vec{x})}{\partial y} + \frac{\partial H(\vec{x})}{\partial z} = 0,$$

$$\frac{\partial A(\vec{x})}{\partial x} + \frac{\partial B(\vec{x})}{\partial y} + \frac{\partial E(\vec{x})}{\partial z} = 0.$$

They have the form of the system (2) and admit the reduction to the KP-equation (4).

References

[1] V.Dryuma: Eight-dimensional the Ricci-flat Riemann space related with the KP-equation. *arXiv:0810.0346 v1 2 Oct 2008, 1–5, v2 15 Oct 2008 1–5.*

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[3] B. Konopelchenko: Quantum deformations of assiciative algebras and integrable systems. *arXiv:0802.3022 v2 [nlin.SI] 28 Apr 2008* 1–24.

[4] V.Dryuma, L. Bogdanov : On nonlinear equations connected with six-dimensional Plebanski space. *arXiv:0812.1637 v1 [physics.gen-ph] 9 Dec 2008* 1–11.