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Non-dispersive traveling waves in strongly inhomogeneous water channels

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Traveling waves are usually studied for media with constant parameters along the wave path. Well known solutions exist for most integrable or near-integrable equations within nonlinear wave physics, including large-amplitude water waves. If parameters of the medium vary slowly along the wave path, solutions in the traveling wave form with variable amplitude and phase can be obtained with the use of asymptotic methods (WKB approach for linear waves, perturbation soliton theory, etc). The amplitude of both linear and nonlinear waves satisfy the conservation of energy flux. It has been pointed out in literature that for certain conditions of parameters of the medium, an asymptotic WKB solution for a linear monochromatic wave coincides with an exact solution even for rapidly varying media. It has been shown in literature that traveling waves in general form can be found within the framework of the variable-coefficient linear wave equation using direct methods and Lie algebra. Concerning nonlinear waves in strongly inhomogeneous media, the number of such exact traveling wave solutions is very limited. One of the examples here is the solution of the variable-coefficient nonlinear Schrodinger equation, which has been found in the form of a soliton with constant amplitude due to a balance between the wave focusing in inhomogeneous media on caustics and wave defocusing, related to the phase chirp. The interest to the problem of traveling waves is related to the possibility of reflectionless transfer of wave energy over long distances.

Here we consider the water wave dynamics in a narrow channel of the parabolic cross-section with variable depth along longitudinal direction and discuss traveling wave solutions in both linear and nonlinear cases.

Governing equations for describing long waves in narrow channels consist of the hyperbolic 1D nonlinear-shallow water system

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + \frac{2H}{3}\frac{\partial u}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial H}{\partial \eta} = g\alpha, \tag{2}$$

where $H = h(x) + \eta$ is the total depth along the longitudinal channel axis, $h(x)$ is unperturbed depth, η is the water displacement, u is depth-averaged velocity,

g is gravity acceleration, and $\alpha(x)$ is the variable bottom gradient along the channel axis.

Various solutions of this hyperbolic system are described in given paper.

1. If the wave amplitude is small ($\eta << h$) and the bottom slope is small ($\alpha << 1$), the asymptotic solution in the framework of the WKB approach is found

$$\eta = A(x)\exp\{i\omega[t - \tau(x)]\}, \quad A(x) \sim h^{-1/2}(x), \quad \tau = \int dx/\sqrt{2gh/3}. \qquad (3)$$

2. It is proved that the solution (3) is a rigorous solution of the linear version of the system (1)-(2), when $h(x) = \alpha x$ even for not small α.

3. The exact solution of the initial nonlinear hyperbolic system is obtained with the use of the hodograph transformation for the case $h(x) = \alpha x$. It presents the superposition of the two nonlinear traveling waves non interacted between them. Such waves break in the singular point $x = 0$.

The main results can be presented as follows:

i) The traveling waves may exist in strongly inhomogeneous media within linear and nonlinear theory. Such solutions are obtained for long water waves propagating in a uniformly inclined narrow channel with a parabolic cross-section. In the linear case they are found directly, and in the nonlinear case the hodograph transformation is used. It is shown that both linear and nonlinear traveling waves of water displacement in the channel should have a sign-variable shape for the wave energy to be bounded.

ii) Traveling waves of water displacement conserve their shape in linear theory. Their amplitudes vary according to the Green's law, which follows from the energy flux conservation, without any limitations of the bottom slope. Traveling waves of water velocity do not conserve their shape even in linear theory and can be described by the superposition of two traveling waves of different amplitudes.

iii) In the nonlinear case traveling waves propagating onshore become steeper with distance and always breaks near the shore. At the same time waves propagating offshore do not break. The variation of the wave amplitude with a distance, which represents the "nonlinear" Green's law, differs from the prediction of the linear theory. The negative wave amplitude grows faster than positive amplitude in shallow water, demonstrating that the energy flux conservation in the nonlinear wave is determined by the total depth, which is less under the wave trough than under the wave crest. The travel time of a nonlinear wave is described well by the linear theory.

iv) It is shown that traveling waves do exist in strongly inhomogeneous media in both, linear and nonlinear cases. These waves propagate over large distances without reflection. The obtained solutions can be useful for detailed analyses of the wave dynamics in channels including processes of wave shoaling, runup and wave breaking.

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