

# Appearance of a topological first integral in a multivalued Poisson dynamical system

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## 1 General remarks

Poisson dynamical systems play a fundamental role and are widely studied in both the classical and quantum mechanics environments. Nevertheless so far very little has been done in the case when multivalued first integrals appear. This phenomenon can take place when the Poisson manifold  $(M, \{, \})$  on which the dynamical system is defined has a non-trivial topology. Note indeed that in the expression of the Poisson bracket of two functions  $f$  and  $g$  only their derivatives appear, i.e.  $\{f, g\}$  is actually a function of the gradients  $df$  and  $dg$ , so it there is no obstruction in evaluating the bracket also with closed differential 1-forms  $\omega$ . The components of any such  $\omega = \omega_\alpha dx^\alpha$  behave exactly like the gradient of a function (i.e.  $\partial_\beta \omega_\alpha = \partial_\alpha \omega_\beta$ ) and indeed locally  $\omega$  is the gradient of a function; the problem is that, unless  $\omega$  is exact (i.e. unless its integral over all non-trivial loops of  $M$  is zero), when one tries to extend to the whole  $M$  any function  $f$  for which locally  $\omega = df$  there are closed paths  $\gamma$  on  $M$  such that, after completing a loop on  $\gamma$  based at a point  $p$ , the value of  $f$  at  $p$  is different from the one it had there at the beginning – i.e.  $f$  is not a well-defined function but rather a *multi-valued* function.

This is the case, for example, when  $M$  is the three-torus  $\mathbb{T}^3$  (with coordinates  $(\theta^\alpha)$ ,  $\alpha = 1, 2, 3$ ) and the Poisson bracket is given by the so-called *magnetic bracket*  $\{f, g\}_H = \epsilon^{\alpha\beta\gamma} \partial_\alpha f \partial_\beta g H_\gamma$ , where  $\epsilon^{\alpha\beta\gamma}$  is the Levi-Civita tensor and  $H = H_\alpha d\theta^\alpha$  a fixed constant 1-form (“magnetic field”). Indeed  $\{, \}_H$  has the obvious Casimir  $h(\theta^\alpha) = H_\alpha \theta^\alpha$  (since  $dh = H$ ) but clearly  $h$  is not single-valued on  $\mathbb{T}^3$  since the angles are multivalued.

This Poisson dynamical system appeared for the first time in the Fifties [5] in the semiclassical theory of the motion of quasi-electrons in normal metals at low temperature under a strong magnetic field and was used to explain the highly anisotropic behaviour of the magnetoresistance tensor. Indeed in such physical environment the current tends to travel in just one direction (for a fixed magnetic field) and this direction is dictated by the asymptotics of the orbits of quasi-electrons, i.e. of the solutions of

$$\dot{\theta}^\alpha = \{\theta^\alpha, F\}_H \tag{1}$$

where the Hamiltonian  $F$  is the Fermi Function of the metal. The rich topological structure beneath it though remained hidden until Eighties, when S.P. Novikov became interested in it as an application of his extension of Morse theory to multivalued functions [6]. The main result of the work of Novikov and its pupils [8, 4, 7, 2] is the existence of a topological first-integral which dictates the asymptotic direction of the solutions of (1). Moreover, in interesting cases the dependence of this first-integral from the direction of  $H$  is fractal.

## 2 Plan of the talk

In this talk we briefly introduce the concept of multivalued Poisson dynamical system and illustrate the concrete case of  $(\mathbb{T}^3, \{, \}_H)$ .

In particular we show how and why the asymptotics of the solutions of (1) are related to a “hidden” topological first-integral. The key point is that, because of the topology of the 3-torus, for a generic direction of  $H$  the open solutions of (1) are bound, on the 2-dimensional energy level  $F_e = \{F = e\}$ , on a component of genus 2 no matter how high is the genus of  $F_e$  (which can be in principle arbitrarily high – note that this behaviour is the opposite with respect to a generic Hamiltonian dynamical system on a surface, when open leaves are dense on the whole surface). This genus-2 component identifies an integral 2-cycle in  $\mathbb{T}^3$  whose homology class in  $H^2(\mathbb{T}^3, \mathbb{Z})$  is the first-integral above.

Finally we show the numerical and analytical results we found for some “toy-model” Hamiltonian [1] and some physical one [3]

## References

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