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Dynamical Systems in the Complex Domain

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1 Introduction

Complex analysis plays a useful role in helping one to understand the mathematical structure of systems that are defined in the real domain. For example, complex analysis provides a simple explanation of the fundamental theorem of algebra and of the region of convergence of Taylor series. It is for this reason that we have pursued a program of extending into the complex domain the dynamical systems that are ordinarily defined and studied in the real domain. We begin this talk by showing how to extend classical dynamical systems into the complex domain. In doing so we discover a rich and interesting class of behaviors. We find that classical systems in the complex domain can exhibit remarkable properties, such as tunneling and band structure, that are normally thought to be limited to quantum systems. It is also hoped that complex analysis of classical systems will eventually provide a deeper understanding of such phenomena as chaotic behavior. In the second part of this talk we discuss the complex analysis of quantum-mechanical systems. Extending a quantum system into the complex domain must be done very carefully because it is essential to preserve the reality of the energy spectrum and the unitarity of time evolution (conservation of probability). When this is done properly, we discover new kinds of quantum theories having unusual properties. In the past few months some of these new quantum systems have been replicated and observed in laboratory experiments, and more experiments are being planned and developed by various groups around the world.

2 Complex Classical Mechanics

To extend classical mechanics into the complex domain one can begin by keeping the (conserved) energy real, but allowing for complex as well as real solutions to the dynamical equations of motion [1]. The solutions that we obtain have a rich and interesting structure; the particles may visit visit multiple sheets of a Riemann surface. Furthermore, the periods of closed orbits may depend on the parameters of the Hamiltonian in a very complicated fashion, behaving differently for rational and for irrational values of a parameter. Next, one may allow the energy to take on complex values (arguing by the energy-time uncertainty principle in quantum mechanics that there is always some uncertainty in the value of the energy, and that this uncertainty may well have a complex component). If we allow the energy to be complex, we find that the classical orbits now allow for particles to exhibit features, such as tunneling and bands and gaps, that are normally associated with quantum systems [2]. In addition, we find interesting behaviors when we examine the complex trajectories in chaotic systems [3].

3 Complex Quantum Mechanics

Complex quantum systems are described by Hamiltonians that are not Dirac Hermitian; that is, Hamiltonians that are not invariant under combined complex conjugation and matrix transposition. Such Hamiltonians can still have real spectra and generate unitary time evolution if they have an unbroken $\mathcal{PT}$ (space-time reflection) symmetry. Like complex classical systems, such quantum systems also have remarkable mathematical properties. In particular, these systems can have a phase transition from an unbroken phase (where the energy eigenvalues are real) to a broken phase (where the eigenvalues become complex). Recent theoretical papers [4, 5] have argued that it is theoretically possible to observe this $\mathcal{PT}$ phase transition in optics experiments. This has now been achieved in the laboratory [6] in an experimental collaboration with physicists from the Universities of Arkansas, Central Florida, INRS-EMT in Quebec, and Sherbrooke [6]. Further experiments at the University of Heidelberg using molecular beams of helium are being designed [7].

References

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