On asymptotic integrability of perturbed evolution equations

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Integrable evolution equations provide approximate information on small-amplitude solu-tions of complex dynamical systems. The derivation of an evolution equation is based on an as-ymptotic expansion of the dynamical quantities that appear in the equations of the original complex system in powers of a small parameter. When the terms that have been omitted in the deriva-tion of the evolution equation are re-instated, the approximate solution of the resulting perturbed evolution equation is expressed in terms of an additional asymptotic expansion in powers of the same small parameter. The perturbed evolution equation is said to be asymptotically integrable if the expansion of its solution obeys:

(1) Corrections to the solution in orders higher than zero can be written as differential poly-nomials in the zero-order approximation.

(2) The zero-order approximation has the same wave solutions as the unperturbed equation (solitons, fronts). The effect of the perturbation on wave parameters is determined by a normal form, which is integrable order-by-order.

Often, perturbed evolution equations are not asymptotically integrable; requirement no. 1 cannot be satisfied, unless requirement no. 2 is violated. However, requirement no. 2, which is the main ingredient of asymptotic integrability, can be fulfilled, if requirement no. 1 is relinquished in one of the two asymptotic expansions mentioned above. From first order onwards, one may incorporate terms that are not differential polynomials in the zero-order approximation. Such terms can account for the obstacles to asymptotic integrability contained in the perturbation, allowing for the satisfaction of requirement no. 2. Results on how bounded non-polynomial terms can be exploited in either expansion so as to satisfy re-quirement no. 2 for wave

solutions of the perturbed Burgers and KdV equations will be reviewed. The classical shallow-water-layer problem, which is approximated by the KdV equation, is unique. The freedom in the expansion is so great that one can derive the KdV equation with perturbations that do or do not contain obstacles to asymptotic integrability. Moreover, the freedom allows for a renormalized normal form, i.e., a normal form that can be formally truncated beyond first order. Consequently, the wave parameters of the zero-order approximation are updated only by a first-order correction. Compared to other valid expansions, in which wave parameters are corrected in every order of the expansion, this corresponds to a mere rearrangement of terms in the asymptotic series. The calculation has been carried out through third order.

Cases in which both requirements can be satisfied simultaneously are the exception. Already in the case of perturbed ODE, the requirement that the higher-order corrections to the solution can be expressed as polynomials in the zero-order approximation is obeyed in the case of the limited, but extremely important, class of dynamical systems for which the unperturbed part is linear, and the perturbation is autonomous. In all other cases, this requirement often fails.