$Sp(m)$-invariant systems

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We consider a class of integrable PDE's and their discrete analogues which are invariant under the action of the symplectic group $Sp(m)$ on the dependent variables. In contrast to the classes of $U(m)$-invariant systems (*e.g.* vector NLS) and $O(m)$-invariant systems (*e.g.* vector mKdV), only few studies have so far been made at $Sp(m)$-invariant systems. Examples in a slightly generalized form include

$$\frac{\partial u_i}{\partial t} + \frac{\partial^3 u_i}{\partial x^3} + 3 \left[\sum_{1 \le j < k \le M} C_{jk} \Big(\frac{\partial u_j}{\partial x} u_k - u_j \frac{\partial u_k}{\partial x} \Big) \right] \frac{\partial u_i}{\partial x} = 0, \quad i = 1, 2, \ldots, M,$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial^3 u_i}{\partial x^3} + 3 \frac{\partial}{\partial x} \left[\sum_{1 \le j < k \le M} C_{jk} \Big(\frac{\partial u_j}{\partial x} u_k - u_j \frac{\partial u_k}{\partial x} \Big) u_i \right] = 0, \quad i = 1, 2, \ldots, M,$$

$$\frac{\partial^2 v_i}{\partial \tau \partial x} + v_i - \left[\sum_{1 \le j < k \le M} C_{jk} \Big(\frac{\partial v_j}{\partial x} v_k - v_j \frac{\partial v_k}{\partial x} \Big) \right] v_i = 0, \quad i = 1, 2, \ldots, M,$$

$$\frac{\partial u_n^{(i)}}{\partial t} + \frac{u_{n+1}^{(i)} - u_n^{(i)}}{1 + \displaystyle\sum_{1 \le j < k \le M} C_{jk} \big(u_{n+1}^{(j)} u_n^{(k)} - u_n^{(j)} u_{n+1}^{(k)} \big)} + \frac{u_n^{(i)} - u_{n-1}^{(i)}}{1 + \displaystyle\sum_{1 \le j < k \le M} C_{jk} \big(u_n^{(j)} u_{n-1}^{(k)} - u_{n-1}^{(j)} u_n^{(k)} \big)} = 0, \quad i = 1, 2, \ldots, M.$$

Indeed, in the canonical case of coupling constants, i.e. $C_{2j-1\,2k} = -C_{2k\,2j-1} = \delta_{jk}$, $C_{2j-1\,2k-1} = C_{2j\,2k} = 0$, $M = 2m$, these systems are invariant under the action of the symplectic group $Sp(m)$ in standard representation.