Action-phase Ermakov quantization of a nonlinear dynamical system on a torus

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In a pioneering paper about Sommerfeld-Epstein quantization of trajectories on a 2D torus [1] that has been re-discovered by Gutzwiller [2], Einstein pointed out the basic impossibility of quantizing quasiperiodic orbits. The present talk wishes to provide a modern illustration of this paradigm by use of the nonlinear Gross-Pitaevskii (GP) mean-field model of axisymmetric vortex experiments performed in a Rb^{87} alkali Bose-Einstein condensate (BEC) confined in an external cigar-shaped parabolic potential [3]. The corresponding BEC order parameter –or nonlinear eigenstate $\Psi(r, \theta)$ — is described by the Madelung ansatz $\Psi = a(r)exp[i\Phi(r)]$ whose phase $\Phi = S/\hbar$ further assumes separation of variables: $S(r) = R(r) + m\hbar\theta$. The azimuthal circulation number m is assumed integer (here: 0 or 1) in order to avoid a multivalued wavefunction of the variable θ and the talk focuses on the radial part $R(r)$ of the Madelung action S, whose gradiant $p = dR/dr$ defines the radial quantum momentum p. The corresponding amplitude-phase formulation of such a separable quantum system has already been investigated in the linear case by use of the Ermakov-Milne-Pinney (EMP) transform [4]. The role of the Ermakov constant or Wronskian W- has been emphasized in order to define the corresponding quantum trajectories $p(W, r)$. Two fundamental properties which do not depend on the value of W have been shown in this linear case (e.g. the 1D harmonic oscillator or the 3D Kepler system), namely : i) the value of the energy levels (which are thus infinitely degenerated with respect to the value of W), and ii) along any closed loop, the following action quantization rule for the conjugate variables p, q holds : $\oint pdq = (n + 1)h$, where the positive integer n is the number of nodes of the eigenstate. Hence the ground state action $\oint pdq = h$ for $n = 0$ which outlines the basic difference of this exact EMP description with respect to the corresponding semiclassical Wentzel-Kramers-Brillouin approximate value $\frac{1}{2}h$. The situation radically changes when such a nonlinearity as the GP one is introduced in the quantum system. Then the linear degeneracy of the energy levels with respect to W is lifted and the quantization of action, namely $\oint pdq = (n+1)h$, yields for a discrete set W_1, W_2, W_3 of values of W a new hyperfine action-phase quantization of energy. It is believed that the corresponding trajectories $p(W, r)$ are those periodic orbits whose quantization has been predicted by Einstein in his 1917 paper [1], while the rest of trajectories $p(W, r$ when $W \neq W_1, W_2, W_3$ would correspond to the unquantizable quasiperiodic orbits, according to Einstein. Such conclusions may be of interest in the analysis of the rotating BEC experiment [3].

References

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